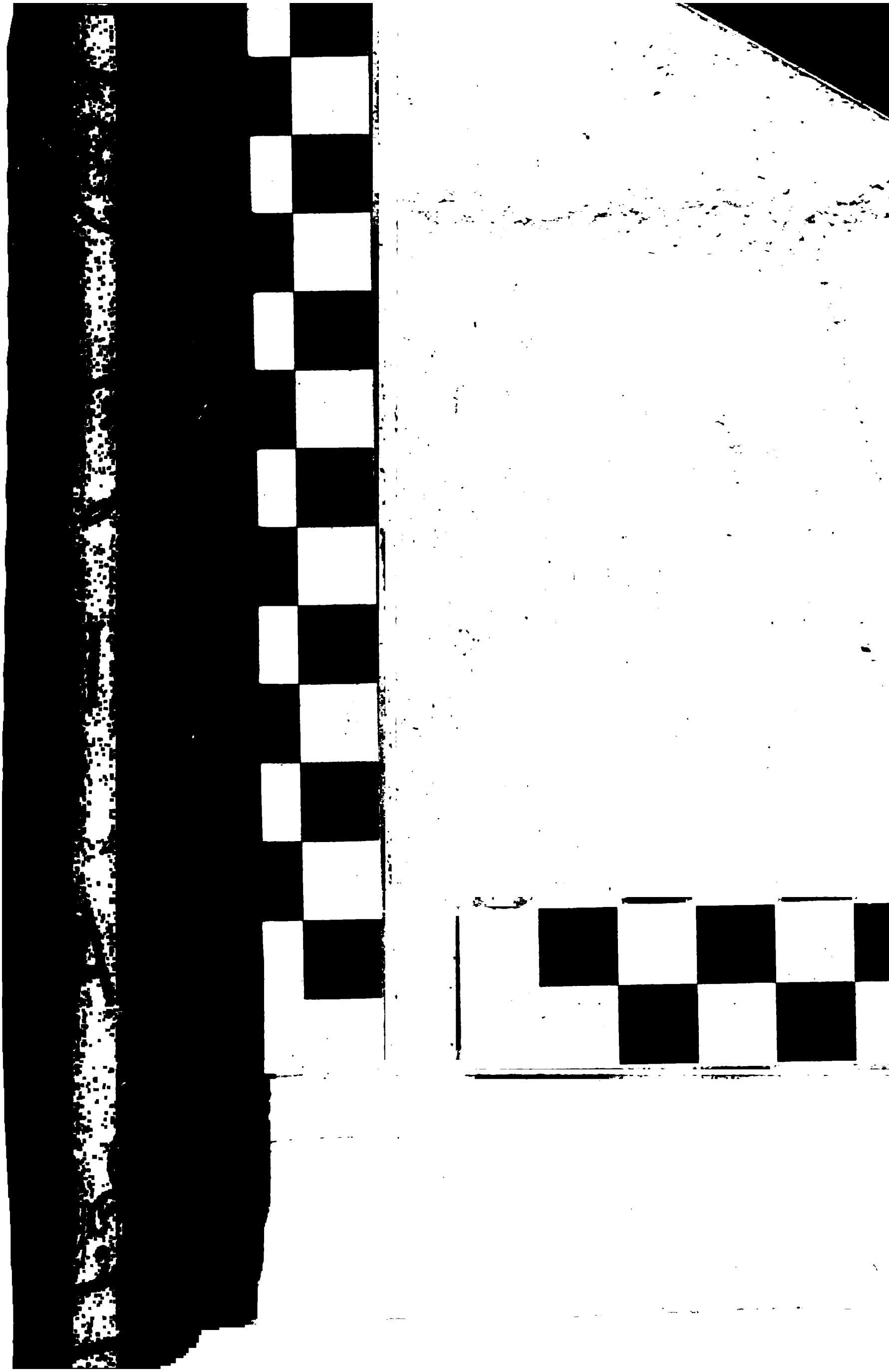


FIRST PRINCIPLES

OF

ALGEBRA.

AN INTRODUCTION TO THE AUTHOR'S COMPLETE, OR UNIVERSITY ALGEBRA.

BY

EDWARD OLNEY,

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF MICHIGAN, AND AUTHOR OF A SERIES OF MATHEMATICAL TEXT-BOOKS.

SHELDON & COMPANY,

NEW YORK & CHICAGO.

OLNEY'S NEW SERIES

EMBRACES THE FOLLOWING BOOKS.

FIRST LESSONS IN ARITHMETIC, } **Two-Book Series.**
PRACTICAL ARITHMETIC. }

This Series has more examples, and at less price, than any ever published.

SCIENCE OF ARITHMETIC.

(*A strictly High School Text-Book.*)

Send for full Circular of OLNEY'S ARITHMETICS.

FIRST PRINCIPLES OF ALGEBRA.

COMPLETE ALGEBRA. (Newly electrotyped in large type.)

NEW GEOMETRY. (In preparation.)

P. of A.

PREFACE

THIS little book is, as its title imports, a mere Introduction to Algebra. It is its purpose to make the transition from the Arabic Notation and Common Arithmetic, to the Literal Notation and Algebra, as simple and attractive as the nature of the subjects will allow. It can be studied by quite young pupils who have but a very elementary knowledge of Arithmetic. It will be found adapted to such of our public schools as wish to introduce the subject of Algebra before the pupil has sufficient maturity to enter upon the COMPLETE SCHOOL ALGEBRA, and for Colleges having a Preparatory Department and desiring some simple introduction to the Author's UNIVERSITY ALGEBRA.

In order to economize space and time, as well as to lead the pupil to feel that he is not entering an entirely new field, some of the more elementary definitions, common to arithmetic and algebra, have been omitted. Nevertheless, great care has been taken not to omit any which could by any possibility be unfamiliar, or which need a more accurate or comprehensive statement than is commonly given.

The order of arrangement is rather that which the pupil can pursue with the greatest ease, than that which a rigid scientific analysis of the subject demands.

In the first sections the topics are approached by the simplest inductions, the rules are preceded by illustrative examples, and followed by explanations and statements of reasons in a free and somewhat colloquial style. But, as the subject proceeds, a gradual transition is made to a more condensed and formally scientific treatment. In a few instances, *processes* have been given and the formal demonstration withheld, though never without apprising the pupil of the fact. It is the purpose of the book to lead the young to comprehend and appreciate mathematical reasoning, as well as to solve problems.

Formal statements of principles, definitions, and rules, when repeated in different members of the series of which this book forms a part, are given in exactly the same language.

A glance at the Table of Contents will inform the reader as to the scope of the book. The elements of Literal Arithmetic, Simple Equations with one, two, and three unknown quantities, Quadratic Equations with a few cases of Simultaneous Quadratics, and Ratio and Proportion, are the principal subjects treated.

Trusting that the book may be a means of interesting the young at the threshold of this great department of mathematical science, and may prove serviceable to the teacher in his efforts to lead his pupils to *think*, as well as "cipher," the author submits it to the judgment of his fellow laborers.

EDWARD OLNEY.

UNIVERSITY OF MICHIGAN,
Ann Arbor, *August*, 1874.

Contents

SECTION XIX.

SECTION XX.

SECTION XXI.

SECTION XXII.

SECTION XXIII.

SECTION XXIV.

SECTION XXV.

SECTION XXVI.

SECTION XXVII.

SECTION XXVIII.

INTRODUCTION TO ALGEBRA,

SECTION I.

HOW LETTERS ARE USED TO REPRESENT NUMBERS.

1. THREE times 5, and 4 times 5, and 2 times 5, make how many times 5?

Three times 7, and 4 times 7, and 2 times 7, are how many times 7?

Three times *any number*, and four times *the same number*, and 2 times *the same number*, are how many times that number?

2. Two times 8, plus 5 times 8, plus 3 times 8, plus 1 time 8, are how many times 8?

Two times 17, plus 5 times 17, plus 3 times 17, plus 1 time 17, are how many times 17?

Two times *any number*, plus 5 times *the same number*, plus 3 times *the same number*, plus 1 time *the same number*, are how many times that number?

3. Five 23's, plus 4 23's, plus 11 23's, are how many 23's? *Ans.*, 20 23's.

1. In Algebra *we often use letters to represent, or stand for, numbers.*

The following exercises will show how:

4. Suppose a stands for some number, as, in the first exercise, for the 5; 3 times a, and 4 times a, and 2 times a, make how many times a?

Again, suppose a stands for some number, as 7 in the first exercise; 3 times a, and 4 times a, and 2 times a, are how many times a?

Again, suppose a stands for *any number*, only that it shall mean *the same number* each time; 3 times a, and 4 times a, and 2 times a, make how many times a?

5. If m stands for (represents) some number; how many times m, are 2 times m, plus 5 times m, plus 3 times m, plus 1 time m?

Does it make any difference what number m stands for, so that it means the *same number* all the time?

Compare this with Ex. 2.

6. Suppose b represents some number (meaning the same number all the time *in this exercise*), 5 b's, plus 4 b's, plus 11 b's, are how many b's?

Compare with Ex. 3.

2. Thus we may use any letter *to represent any number, provided it always means the same number in the same exercise or problem.*

3. When a letter is used *to represent a number, the figure which tells how many times the number represented by the letter is taken, is simply written before the letter, the word "times" being left out.*

Thus $3a$ means 3 *times* a, $4b$ means 4 *times* b, $7m$ means 7 *times* m, $105x$ means 105 *times* the number represented by x, whatever that number may be.

4. **The number** *placed before a letter to tell how many times the letter is taken, is called a* **Co-efficient.**

If no figure stands before a letter, the letter is taken *once*, or its co-efficient is said to be 1.

Thus, m means one time m.

7. How many times the number represented by b, are $4b$, $3b$, $6b$, and b?

That is, 4 times *some number*, plus 3 times *the same number*, plus 6 times *the same number*, plus one time *the same number*, are how many times that number?

8. $5a$, plus a, plus $6a$, plus $8a$, are how many times a?

Ans., $20a$.

Query.—If the a in $5a$ meant one number, the a alone *another* number, the a's in $6a$ and $7a$ still other numbers, could you answer this exercise in the same way? You could not answer it at all. The a must mean the same number all the time, in the same example.

9. $10a+5a+7a+2a$, are how many times a?

Query.—Is it necessary that a should mean the same thing in this exercise that it did in Ex. 8?

10. $3a+2a+5a+8a$, are how many times a?

Ans., $18a$.

How much is this if a is 6? *Ans.*, 108.

How much if a is 11? *Ans.*, 198.

11. Eleven times 8, minus 5 times 8, are how many times 8?

12. $11a-5a$ are how many times a?

Eleven times any number, minus 5 times the same number, are how many times that number?

13. $12x-7x=$how many times x?

How much is this if x represents 3?

If x represents $2\frac{1}{5}$? *Ans.* to the last, 11.

14. $5b+4b+10b-12b=$how many times b?

15. How much is $3m+8m-4m+6m-5m-2m$?

16. What is $10a$ divided by 2; that is, what is $\frac{1}{2}$ of $10a$?

$27x$ divided by 9 is how much? *Ans.* to the last, $3x$.

17. How many times a number is 10 times that number, divided by 2; that is, $\frac{1}{2}$ of 10 times a number?

18. How much is $\frac{1}{8}$ of $48x$?

$25x$ divided by 5?

$\frac{1}{11}$ of $11x$?

$11x$ divided by 11?

$7x$ divided by 7?

19. Divide $10x$ by 5, then add $3x$, then multiply by 2, then subtract $4x$, then divide by 3. What is the result?

20. Multiply $2a$ by 3, then subtract a, then multiply by 4, then divide by 2, then by 5, then add $3a$. What is the result?

[NOTE.—The teacher should extend such exercises until his pupils can perform them mentally, as rapidly as one would naturally pronounce them.]

SECTION II.

LETTERS WRITTEN SIDE BY SIDE.

6. When letters representing numbers ***are written side by side, as in a word, their product is indicated.***

Thus, ab means the product of the two numbers represented by a and b.

$3abc$ means 3 times the product of the numbers represented by a, b, and c.

[NOTE.—Instead of saying, as above, *the number represented* by a, we usually say "the number a," or, "a," without using the word number at all. Thus we say 3 times the product of a, b, and c.]

1. If $a=3$, and $b=2$, what is ab? *Ans.*, $ab=6$.

2. If $m=5$, and $n=8$, what is mn?

3. What is cd, if $c=4$, and $d=8$?

4. What is cd, if $c=7$, and $d=2$?

5. What is cd, if $c=10$, and $d=2\frac{1}{2}$?

6. What is cd, if $c=\frac{2}{3}$, and $d=\frac{3}{4}$?

7. What is amn, if $a=5$, $m=7$, and $n=2$?
Ans., $amn=70$.

8. What is amn, if $a=2\frac{1}{2}$, $m=4$, and $n=3\frac{1}{5}$?
Ans., $amn=32$.

9. What is amn, if $a=\frac{2}{5}$, $m=\frac{3}{4}$, and $n=\frac{1}{2}$?

10. What is $3ab$, if $a=2$, and $b=4$? *Ans.*, $3ab=24$.

11. What is $3ab$, if $a=5$, and $b=4$?

12. What is $3ab$, if $a=20$, and $b=7$?

13. What is $12abc$, if $a=2$, $b=3$, and $c=11$?

Ans., $12abc=792$.

14. What is $12abc$, if $a=1$, $b=7$, and $c=2$?

15. What is $12abc$, if $a=2$, $b=2$, and $c=2$?

16. What is $12abc$, if $a=\frac{2}{3}$, $b=\frac{1}{2}$, and $c=\frac{3}{4}$?

17. What is $7mn$, if $m=\frac{2}{3}$, and $n=\frac{3}{5}$? *Ans.*, $7mn=2\frac{4}{5}$.

18. What is $11amn$, if $a=2\frac{1}{2}$, $m=3\frac{2}{3}$, and $n=5$?

19. What is $13abc$, if $a=\frac{2}{7}$, $b=\frac{3}{5}$, and $c=\frac{1}{11}$?

20. What is $23mn$, if $m=3$, and $n=10$?

21. What is $23mn$, if $m=132$, and $n=2548$?

SECTION III.

EXPONENTS AND TERMS.

7. If we want to represent the product of a number *represented by a letter, as* **a**, *by itself a certain number of times, instead of writing* **aa**, *or* **aaa**, *etc., as we might, we write* $\mathbf{a^2}$, $\mathbf{a^3}$, *etc.*

Thus b^4 means the same as $bbbb$. a^2 is read "a square;" a^3, "a cube;" b^4, "b fourth power;" x^5, "x fifth power," etc.

1. Read m^2. What does it mean? How otherwise could you write it?

2. Read x^3. What does it mean? How otherwise could you write it?

3. If I wish to write the product of b taken 4 times as a factor, how do I do it?

4. Write $aaaa$ in the proper form?

5. Indicate the product of y taken 5 times as a factor?

6. What is a^3 if $a=2$? *Ans.*, $a^3=8$.

7. What is m^4 if $m=3$?

8. What is b^2 if $b=\frac{2}{3}$?

9. What is b^3 if $b=1\frac{1}{2}$?

8. The little figure *placed at the right and a little above the letter is one form of what is called an* **Exponent.**

The pupil must not get the idea that all exponents mean just what has now been explained.

This is the case only when the exponent is a whole number without any sign, or with the + sign.

Thus a^{-3} does not mean aaa. Nor does $a^{\frac{2}{3}}$ mean any such thing, although the -3, and the $\frac{2}{3}$ are exponents. What these do mean will be explained in SEC. xxiv.

[NOTE.—It is of the utmost importance that the pupil be guarded, from the outset, against the notion that an exponent necessarily indicates a power. This false notion, once in the head, plagues the pupil ever after.]

10. What is $4a^2b$, if $a=2$, and $b=5$?

Suggestion.—Notice that $4a^2b=4\times a\times a\times b$. Hence $4a^2b=4\times2\times2\times5$. Or $4a^2b=4\times a^2\times b$, and as $a=2$, $a^2=4$. Hence $4a^2b=4\times4\times5$.

11. How much is $10a^5c^2y$, if $a=2$, $c=1$, and $y=3$?

12. How much is $a^2b^2+2ay-by$, if $a=4$, $b=3$, $y=2$?

How much is $3a^2by^2-2ay^2+5b$, the letters having the same values as before ?

How much $5by-2ab^3+4a^2y^2-2a$?

13. How many times ab^2 is $4ab^2+2ab^2-3ab^2$?

How many times a^3y is $10a^3y+4a^3y-6a^3y-a^3y$?

14. How many times am^2y^3 is 4 times $3am^2y^3$?

How much is 6 times $2am^2y^3$?

Four times $7a^2b^2c^2$?

Ten times $13mn^2x^3$?

15. How many times ax is $\frac{1}{5}$ of $20ax$?

$\frac{1}{7}$ of $35ax$?

$102ax$ divided by 3 ?

How much is $\frac{1}{8}$ of $72a^2x^3$?

$125x^2y^2$ divided by 25 ?

$18ny^3$ divided by 9 ?

---

9. We have learned in arithmetic *that representing numbers by the figures 1, 2, 3, 4, etc., is called Arabic or Decimal Notation.*

In like manner, representing numbers by the small letters of the alphabet, as a, b, c, d, - - - - x, y, etc , is called ***Literal Notation.***

The pupil will see that this *Literal Notation* is altogether a different thing from the *Roman Notation*, in which the seven capital letters, I, V, X, L, C, D, M, are used.

Because the *Literal Notation* is so much used in Algebra, it is often called the ***Algebraic Notation.*** But this notation is just as much used in some other branches of Mathematics, as in Algebra.

---

10. An expression *like* $7a^2x$, *not separated into parts by the signs* + *or* −, *is called a* **Term,** *or a* **Monomial.**

If there are *two* such terms joined together by either of the signs + or −, the two taken together are called a ***Binomial,*** as $6b^2x^2+2ay^3$, or $10x-3ay$.

If three terms are joined in this way it is called a ***Trinomial,*** as $3a^2y-2ab+21x$.

Any expression consisting of more than one term is, in general, called a ***Polynomial.***

Ex. Point out the monomials, binomials, trinomials, and polynomials, in the following: $2ax-3b^2$, $5xy-6cd+a-2y$, $3a^2m^2xy$, c^2-d^2, $a+m$, $a+b+c-d$, $225a^2b^2c^2d^2$, $abcd$, $a-b$, ab, $c-x^2y+ax$, x^2+y^2, $10a^2+3xy$.

---

11. Terms which have the same letters, *affected with the same exponents, are called* **Similar.**

Thus $12a^2y$, $6a^2y$, and $-3a^2y$ are similar; but $12ay$, $6a^2y$, $3cx$, are not similar.

Ex. Point out the similar terms among the following: $3a^2x$, $2ax$, $-5a^2x^2$, ax, $17ax^2$, $16cy^2$, $-12c^2y^2$, $3a^2x^2$, $-5cy^2$, $6cy^2$, $10ax^2$, c^2y^2.

12. Terms having the + sign *are called* **Positive,** *and those having the − sign,* **Negative.** *If no sign is written before a term the sign + is understood.*

SECTION IV.

EXAMPLES FOR PRACTICE IN USING LETTERS TO REPRESENT NUMBERS.

13. If I buy *5 pencils at 8 cents apiece, they cost me 5 times 8 cents,* **i. e.,** *5 × 8 cents.*

So, if I buy a pencils at b cents apiece, they cost me a times b cents, *i. e.,* ab cents.

Now we notice that in the first case we can tell more particularly how much the pencils cost, since $5 \times 8 = 40$. But in the second case we can only say that the pencils cost a times b, or ab cents, since we do not know just what numbers are represented by a and b.

1. John bought m oranges at n cents each. How much did they cost him? If m means 3, and n means 5, how much is it?

2. Mary buys n yards of calico at x cents per yard. How much does it cost her?

3. Henry buys 5 packages of pencils. There are a pencils in each package; and the pencils cost him x cents each. How much does he pay for the whole?

Ans., $5ax$ cents.

4. If 7 jugs are filled with molasses, each jug holding m quarts, and a quart is worth y cents, how much is the whole worth?

5. A boy sold 5 quarts of nuts for 4 cents per quart, 8 quarts of berries at 6 cents per quart, and 3 melons at 10 cents each. He received

$$5 \times 4 + 8 \times 6 + 3 \times 10 \text{ for the whole.}$$

Show in like manner what would be received for m quarts of nuts at n cents per quart, a quarts of berries at b cents per quart, and c melons at d cents each.

6. Represent the value of m yards of cloth at x dollars per yard, n yards at y dollars per yard, and a yards at z dollars per yard.

7. A merchant had 5 rolls of cloth containing c yards each, 7 rolls with m yards each, and 12 rolls with n yards each. The first kind was worth a dollars per yard, the second b, and the third d. What polynomial represents the value of the whole?

8. If I buy 6 barrels of apples at 2 dollars per barrel, and pay for it with cloth at 3 dollars per yard, I must give $\frac{6 \times 2}{3}$ yards of cloth. Express in a similar manner the number of yards I must give if I buy a barrels at m dollars each, and pay in cloth at n dollars per yard.

9. Represent 5 times the square of a divided by 3 times the cube of b.

Suggestion.—Division is represented, as in arithmetic, by writing the divisor under the dividend with a line between them. Hence we have $\frac{5a^2}{3b^3}$.

10. Represent 10 times a multiplied by the square of b, divided by 11 times the product of the cube of m and the 4th power of n.

12. Represent the binomial 3 times the product of a and b plus 5 times the cube of x, divided by the binomial 2 times m minus the square of n.

ADDITION

SECTION V.

Of Similar Terms.

Ex. 1. How many times am^2 is $4am^2+5am^2-3am^2-am^2$? What are similar terms? (See **11.**) Are the terms in this example similar to each other.

14. Similar terms can be united *into one term, which shall express the aggregate* value of the whole, as we see from this example and like examples in Sec. 1. This aggregate value is called the* **Sum.**

* *Aggregate* means *collected, united.*

2. What is the *sum* of $4ax$, $3ax$, $2ax$, and ax? When no sign is written before a term what sign is understood? (See **12**.)

3. What is the *sum* of $5ax$ and $-2ax$?

Suggestion.—This means the same as, What is the aggregate value of $5ax$ and $-2ax$? Or, How many times ax is $5ax-2ax$?

4. What is the *sum* of $7ay^2$, $-3ay^2$, ay^2, $2ay^2$, and $-4ay^2$? Or, How much is $7ay^2-3ay^2+ay^2+2ay^2-4ay^2$?

Ans., $3ay^2$.

5. What is the *sum* of $11m^2n^2, -4m^2n^2, 6m^2n^2, 2m^2n^2, -5m^2n^2, -m^2n^2$, and $3m^2n^2$?

15. Addition is the process *of combining several quantities, so that the result shall express the aggregate value in the fewest terms consistent with the notation.*

6. *Add* $3ay$, $5ay$, $2ay$, and ay.

7. *Add* $5n^2x$, $-2n^2x$, $-n^2x$, and $7n^2x$.

16. The pupil may think it strange that we should use subtraction in adding, *i. e.*, that we should call the result of putting together $5ax$, and $-2ax$, addition, when the work of uniting them is performed by subtraction. To understand this it is only necessary to keep in mind just what we mean by addition, as we use the term here. We mean by it simply *putting quantities together.* Whether a quantity put with another increases it or diminishes it depends upon its nature. If 2 boys are drawing a sleigh, and another boy puts in his strength, the aggregate result will depend upon *which way* the last boy pulls; if he pulls the *same way* as the others, it will *increase* the effect; but, if he pulls in the *opposite* direction, it will *diminish* the effect. The *aggregate* or *sum* will be greater in the former case when the third boy adds his strength, and less in the latter.

8. Five boys are pulling at a sleigh. 3 of them pull in one direction, $5m$, $2m$, and $7m$ pounds, respectively. The other two pull in the opposite direction, $3m$, and m, respectively. What is the aggregate, or *sum*, of their efforts?

Suggestion.—Suppose we call the efforts to pull the sleigh in the first direction *positive*, and the efforts to pull it in the *opposite* direction *negative*. Our question then is, What is the sum of $+5m$, $+2m$, $+7m$, and $-3m$, and $-m$? Or, What is $5m+2m+7m-3m-m$?

9. I have 4 pieces of property worth respectively $10ax$ dollars, $8ax$ dollars, $6ax$ dollars, and $13ax$ dollars. If now I *add debts* to the amount of $5ax$ dollars, $7ax$ dollars, and $3ax$ dollars, what is the aggregate, or *sum*, of what I am worth? *i. e.*, How much is $10ax+8ax+6ax+13ax-5ax-7ax-3ax$?

---

17. Positive and Negative *are terms primarily applied to concrete quantities which are, by the conditions of a problem, opposed in character.*

ILL.—A man's *property* may be called positive, and his *debts* negative. Distance *up* may be called positive, and distance *down*, negative. Time *before* a given period may be called positive, and *after*, negative. Degrees *above* 0 on the thermometer scale are called positive, and *below*, negative.

10. A speculator made by three trades \$200, \$100, and \$50. He then lost in two trades \$300, and \$250. What was the aggregate, or sum, of the whole? Was it gain, or loss? Shall we call it positive, or negative? Will it be written with the + sign before it, or with the − sign?

11. Three boys are pulling at a sleigh, two, attempting to draw it forward, pull $7m$, and $6m$ pounds, respectively; the other, attempting to drag it backward, pulls $15m$ pounds. Which way does the sleigh move? $7m+6m-15m$ makes how many times m? What sign does the result, or *sum*, have? What does the $-$ sign of the result mean?

12. What is the sum of $5cy^2$, $8cy^2$, $-10cy^2$, cy^2, and $-27cy^2$? Which are in excess, the negative, or the positive quantities, in this example?

18. In adding similar terms, *if the terms are all positive, the sum is positive; if all negative, the sum is negative; if some are positive and some negative, the sum takes the sign of that kind (positive or negative) which is in excess.*

Adding Dissimilar Terms.

1. If I buy an orange for a cents and a knife for b cents, how much do I pay for both?

2. If I have one piece of property worth $7ax$ dollars, and another worth $10my$ dollars, how much are both together worth? (See Examples 5, 6, 7, Sec. IV.)

3. If I travel east $5m$ miles one day, $3n$ the next, and $7a$ the next, what polynomial expresses the aggregate of my travels eastward? Why can we not unite these terms into one term? Can we unite any two of them into one? Why not?

19. Dissimilar terms *are not united into one by addition, but the operation of adding is expressed by* *writing them in succession with the positive terms preceded by the + sign and the negative by the − sign.*

4. What is the sum of $5ax$, $3cy$, $-12a^2x^2$, $5m^2$, and $-3c^2y^2$? Are any of these terms similar? Can any two of them be united into one?

5. What is the sum of $5ax$, $-3y^2$, $3ax$, $-2mn$, $4y^2$, $-6ax$, and cy? What other terms among these are similar to $5ax$? This sum is a polynomial of 4 terms. Why, according to the definition of addition (**15**), is not $5ax+3ax-2mn+4y^2-6ax+cy$ the sum?

Adding Polynomials Containing Similar and Dissimilar Terms.

20. RULE.—*Write the expressions so that similar terms shall fall in the same column. Combine each group of similar terms into one term, and write the result underneath with its own sign. The polynomial thus found is the sum sought.*

1. Add $5ax-2cy$, $3ax+4cy$, $cy-2ax$, $-4ax-3cy$, $-ax+5cy$, and $2ax+2cy$.

OPERATION.

Having written the numbers so that similar terms fall in the same column, we may begin to add with any column we choose. Adding the right-hand column we find it makes $+7cy$, and write this sum underneath the column added. In like manner the other column makes $3ax$ (or $+3ax$), which we write without any sign, as + will then be understood. The sum is $3ax+7cy$.

$$\begin{array}{r}
5ax-2cy\\
3ax+4cy\\
-2ax+\ cy\\
-4ax-3cy\\
-\ ax+5cy\\
2ax+2cy\\
\hline
3ax+7cy
\end{array}$$

Reasons.*

As we wish to unite the terms as much as possible, so that the aggregate value of all the polynomials shall be expressed in the simplest form (fewest terms), according to the definition (15), and as only similar terms can be united into one, we write similar terms in the same column *for convenience.*

Then we find the aggregate of each column of similar terms by (18), and as the sums of the several columns are *dissimilar* terms, we can add them only by connecting them with their respective signs, according to (19).

2.

$$\begin{array}{rrr}
5cd- & 2a+ & 4xy \\
2cd+ & 3a- & 5xy \\
 & 8a- & 2xy \\
-6cd & & +14xy \\
 & -\ 3a- & 7xy \\
 & +11a+ & xy \\
4cd- & 15a & \\
\hline
5cd+ & 2a+ & 5xy
\end{array}$$

3.

$$\begin{array}{rrr}
10am- & 3dy^2+ & 2a^2x \\
-\ 6am+ & 4dy^2- & 10a^2x \\
4am- & 8dy^2- & 2a^2x \\
7am- & 13dy^2+ & 6a^2x \\
-\ 9am+ & 2dy^2- & 5a^2x \\
am+ & 18dy^2- & a^2x \\
\hline
7am & & -10a^2x
\end{array}$$

4. Add $2a+8b-4c-9$ and $5a-3b+2c-10$.

Sum, $7a+5b-2c-19$.

5. Add $3ay+4bx-5ac$, $7bx-3ac+2ay$, $8ac-7ay+2bx$, and $9ac-3bx+7ay$.

6. Required the sum of $3ax-4bc+12cx$, $7cx-5ax+14bc$, $8ax-12bc+3cx$, and $2bc-6ax+8cx$.

* A *Rule* states a process in general terms. Let the pupil always be required to give the *reason* for every step. This should be done in the case of each example.

7. Add $ax-4ab+bd$, $3bd-2ax+ab$, $7ab-2ax-bd$, and $5ab-3ax+12bd$.

8. Add $3abd+4abx-5cx$, $8cx-11abx+12abd$, $9abx-12cx+3abd$, and $7cx-15abx+3abd$.

9. Add $7b-2a^2-xy$, $5a^2-6b+3xy$, $b-3a^2+4$, and $a^2-b-3xy$. *Sum*, $b+a^2-xy+4$.

10. Add $3b^2-2a^2+13$, $3a^2-2b^2-5$, $4ab+7a^2-3$, and $2b^2-a^2+ab$.

11. Add ax^2-2y+b, $2y+2ax^2-3b$, $4ax^2-y-b$, and $2b-3ax^2+b$.

12. Add $2c^2+a^2+3bc$, $5c^2-3a^2-2bc$, c^2+2a^2-bc, and $b+bc-3$.

13. Add $ab+a^2c-5$, $3ab-3a^2c+7$, $2a^2c-2ab-3$, and $ab+a^2c+5$.

14. Add $3b^2-2a^2x+b$, $-b^2+3a^2x-3b$, b^2-a^2x+c, and $3b^2+b-3c$. *Sum*, $6b^2-b-2c$.

15. Add $2a^2+3-ac$, $3a^2-7+ac$, $3ac-5a^2+9$, and $3ac+4-a^2$. *Sum*, $-a^2+9+6ac$.

16. Add b^2c+2-y^2, y^2-3b^2c-10, $2b^2c-3+2y^2$, and b^2-y^2+5.

[NOTE.—For the addition of terms with respect to a common letter, and the use of the parenthesis in general, see Sec. X.]

SUBTRACTION

SECTION VI.

1. If you add $-2ax$ to $5ax$, how much of the $5ax$ will be destroyed?

2. If you add $2ax$ to $-5ax$, how much of the $-5ax$ will be destroyed?

21. Adding a negative quantity *destroys an equal positive quantity; and adding a positive quantity destroys an equal negative quantity.*

3. How can you show that adding $2ax$ to $5ax$ destroys $-2ax$? *Answer.* $5ax$ is the same as $7ax-2ax$. Now, adding $+2ax$* to $7ax-2ax$, the $+2ax$ destroys the $-2ax$ and leaves $7ax$.

4. How can you show that adding $-2ax$ to $-5ax$ destroys $+2ax$?

Suggestion.—Observe that $-5ax$ is the same as $-7ax+2ax$.

5. In subtraction in arithmetic you have learned, that the subtrahend and remainder (or difference) added together make the minuend. If, therefore, you destroy the subtrahend out of the minuend what remains? If $5a+3bx-2cy$ is the subtrahend, what would you have to add to a minuend to destroy this subtrahend out of it? What would you have to add to destroy $5a$? What to destroy $+3bx$? What to destroy $-2cy$? Then, if you

* The + sign is written in such cases simply for emphasis.

add to the minuend $-5a-3bx+2cy$, what would be left? (The Remainder.)

How to Perform Subtraction.

22. RULE.—*Change the signs of each term in the subtrahend from + to −, or from − to +, or conceive them to be changed, and add the result to the minuend.*

1. From $5ax-2cy+10b-8$, subtract $2ax+3cy-12b-5$.

OPERATION.

$5ax-2cy+10b-8$	*Minuend.*
$-2ax-3cy+12b+5$	*Subtrahend* with its signs changed.
$3ax-5cy+22b-3$	*Remainder* obtained by

adding the subtrahend with its signs changed to the minuend.

REASONS.

Why do you change the signs of the subtrahend? *Ans.* To get a quantity which added to the minuend will destroy out of it an amount equal to the given subtrahend, according to (21).

Why do you add the subtrahend with its signs changed to the minuend? *Ans.* Because, as the minuend is the sum of the subtrahend and remainder, if we destroy the subtrahend from out the minuend, we have left the remainder.

2. From $4ab+3c^2-xy$ subtract $2ab-c^2+3xy$.
Rem., $2ab+4c^2-4xy$.

3. From $5a^2+3bc-cd+3x$ subtract $2a^2-bc+3cd$.
Rem., $3a^2+4bc-4cd+3x$.

4. From $7a^2b-abc+4xy$ subtract $a^2b+5abc-xy+mz$.

5. From $8a^3-4a^2b-2bc+10$ subtract $3a^3+a^2b-5$.

6. From $3b^2c+abx+2xy^2$ subtract $b^2c-3abx+2xy^2-3$.

7. From $8ax-3by+4dx$ subtract $3dx-5ax+3by$.

8. From $9bcx+7aby-4bx$ subtract $3bcx+2aby-4bx$.

9. From $3bx-4acy+bxy$ subtract $2acy-5bx-bc$.

10. From $9ab-7de+8eg$ subtract $3eg-7de-9ab$.

11. From $2x^2-3y^2$ subtract $2x^2+3y^2$.

12. From $x^2+2xy+y^2$ subtract $x^2-2xy+y^2$.

13. From $x^2-2xy-y^2$ subtract $x^2-2xy+y^2$.

14. From $a-b$ subtract $a+b$.

15. From $a+b$ subtract $a-b$.

16. From $3x^2-2ay$ subtract $2b^2-3mn$.

Suggestion.—The subtrahend with its signs changed is $-2b^2+3mn$. This added to the minuend gives $3x^2-2ay-2b^2+3mn$, as the remainder. Since there are no similar terms, there are none that can be united into one.

17. From $30x-215y$ subtract $7a^2b^2$.

18. From $a-b$ subtract $x-y$.

19. From $8x^2-2ay$ subtract $2x^2-cd+8$.

Rem., $6x^2-2ay+cd-8$.

20. From $3m^2n^2+6xy-3c^2-10+12f$ subtract $c^2-m^2n^2+6xy-ab+5$.

[**NOTE.—For the subtraction of terms with reference to a common letter, and the use of the parenthesis in general, see Sec. X.**]

Multiplication

SECTION VII.

1. What is the difference between 3 times 4, and 4 times 3?

What then is the difference between *ab* and *ba*?

2. What is the difference between $3 \times 5 \times 7$, and $7 \times 3 \times 5$, and $5 \times 7 \times 3$, and $3 \times 7 \times 5$, and $5 \times 3 \times 7$, and $7 \times 5 \times 3$?

What then is the difference between *abc*, *acb*, *cba*, *cab*, *bac*, and *bca*?

23. The product *of several factors is the same in whatever order they are taken. It is, however, customary to write the literal factors of a term in their alphabetical order.*

Thus we write *ax*, not *xa*; also *abc* instead of any other of the six possible orders.

3. Three times \$5 is 15 what?

Six times 7 rods is 42 what?

What is the product always like?

4. Can you multiply \$5 by \$3?

What would 3 *dollars* times \$5 mean?*

* Of course this is absurd; it is nonsense. The multiplier is always to be conceived as an abstract number when the operation is performed, as has been taught in arithmetic.

Can you multiply 7 gallons by 6 gallons?

What kind of a quantity must the multiplier always be conceived to be?

24. The product *is always to be conceived as of the same kind as the multiplicand, and the multiplier as an abstract (mere) number, showing simply how many times the multiplicand is taken.*

Signs of the Product.

1. Since a multiplied by b is ab, if a is a positive quantity what is ab? If a is a negative quantity what is ab?

Then $+a$ multiplied by b gives what? (*Ans.* $+ab$, according to **24**.)

Also $-a$ multiplied by b gives what? (*Ans.* $-ab$, according to **24**.)

2. If we understand the sign $+$ when placed before a multiplier to indicate that the product is to be *added*, and the sign $-$ that it is to be subtracted, what is $+a$ multiplied by $+b$?

Answer. $+a$ multiplied by b is $+ab$, as we saw above. And, as the sign $+$ before the multiplier b shows that this product is to be added (to any others with which it may chance to be connected), it is to be written with its own sign. Hence $+a$ multiplied by $+b$ is $+ab$.

3. With the same understanding as in Example 2, what is the product of $+a$ by $-b$?

Answer. $+a$ multiplied by b gives $+ab$. But as the $-$ sign before the multiplier shows that this product is to be subtracted (from any other quantities with which it may chance to be connected), *its sign must be changed.* Hence $+a$ multiplied by $-b$ is $-ab$.

4. With the same understanding as above, what is the product of $-a$ by $+b$?

Answer. $-a$ multiplied by b is $-ab$. And as the $+$ sign before the multiplier shows that the product is to be added, it is to be written with its own sign. Hence $-a$ multiplied by $+b$ is $-ab$.

5. With the same understanding as above, what is the product of $-a$ by $-b$?

Answer. $-a$ multiplied by b is $-ab$. But as the $-$ sign before the multiplier shows that this product is to be subtracted, its sign must be changed. Hence $-a$ multiplied by $-b$ is $+ab$.

25. When two factors *have the same sign their product is positive: when they have different signs their product is negative.*

6. What is the product of $-a$, $-b$, and $-c$?

Suggestion.—The product of $-a$ and $-b$ is $+ab$; and the product of $+ab$ and $-c$ is $-abc$.

7. What is the product of $-a$, $-b$, $-c$, and $-d$?

Suggestion.—The product of $-a$ and $-b$ is $+ab$; the product of $+ab$ and $-c$ is $-abc$; and the product of $-abc$ and $-d$ is $+abcd$.

26. The product of any number *of positive factors is positive. But the product of an* **odd** *number of* **negative** *factors is* **negative,** *whereas the product of an* **even** *number of* **negative** *factors is* **positive.**

To Multiply Two Monomials Together.

1. What is the product of $3ab$ and $5ab$?

Answer. Since in the literal notation a product is indicated by writing the quantities side by side (Sec. II.), $3ab \times 5ab$ may be written $3ab5ab$. Now, as the order of the factors is immaterial (**23**), this may be written $3 \times 5aabb$. Again $3 \times 5 = 15$, $aa = a^2$, and $bb = b^2$. Hence $3ab \times 5ab = 15a^2b^2$.

27. RULE.—*Multiply the numerical co-efficients as in the decimal notation, and to this product affix the letters of both the factors, affecting each with an exponent equal to the sum of the exponents of that letter in the factors. The sign of the product will be + when the signs of the factors are alike, and − when they are unlike.*

2. Multiply $5my$ by $3mx$. *Prod.*, $15m^2xy$.

3. Multiply $8ay$ by $7a^2y^3$. *Prod.*, $56a^3y^4$.

4. Multiply $4a^2b^2$ by $6a^2b^2$.

5. Multiply $10a^2x^3y$ by $7cxy^2$.

6. Multiply $3a$ by $2b$.

7. Multiply $-5a^2$ by $4a$. *Prod.*, $-20a^3$.

8. Multiply $7ab$ by $-5a^2x$. *Prod.*, $-35a^3bx$.

9. Multiply $-31a^2y$ by $-10ay^2$. *Prod.*, $310a^3y^3$.

10. Multiply $-15m^2ny$ by $3n^5$.

11. Multiply $-my$ by $+my$.

12. Multiply $-a^2b^2$ by $-a^3b^3$.

13. Multiply acy^2 by $-3ac$.

14. Multiply x^2 by $-x^3$.

15. Multiply together $3ax$, $-5a^2$, and $2ax^2$.

Suggestion.—What is the product of $3ax$ and $-5a^2$? This product multiplied by $2ax^2$ gives what?

16. Multiply together $3c^2y$, $4cy^3$, $-acy$, and $-2a$.

17. Multiply together $74c^3y^2$, $217ac^2$, and $-43b^2y^3$.

To Multiply two Factors together when one or both are Polynomials.

1. Multiply $3x-2y+4z$ by $5x+3y-2z$.

OPERATION.

$$\begin{array}{l}
3x-2y+4z \\
\underline{5x+3y-2z} \\
15x^2-10xy+20xz \\
\qquad\quad +\ 9xy \qquad\quad -6y^2+12yz \\
\qquad\qquad\qquad -\ 6xz \qquad\quad +\ 4yz-8z^2 \\
\hline
15x^2-\quad xy+14xz-6y^2+16yz-8z^2
\end{array}$$

EXPLANATION.—I am to take $3x-2y+4z$, $5x+3y-2z$ times, since this is what is meant by multiplying by $5x+3y-2z$. (See definition of multiplication in arithmetic.)

As a matter of convenience I write the multiplier under the multiplicand. Then I multiply the multiplicand by $5x$. This is done by multiplying the parts (terms) of the multiplicand separately, and adding the result; that is, connecting them by their own signs. Thus I find that $5x$ times $3x-2y+4z$ is $15x^2-10xy+2xz$.*

Again, taking the multiplicand $3y$ times I have $9xy-6y^2+12yz$. If this is added to $15x^2-10xy+2xz$ the sum will be $5x+3y$ times $3x-2y+4z$. So I write it under the first partial product, as in addition, but do not add it till I have all the partial products.

I have now taken the multiplicand $5x+3y$ times. But this is $2z$ *too many* times, since I was to take it $5x+3y$ *minus* $2z$ times. Hence I am to take $2z$ times the multiplicand and *subtract* it from the former result. This I do by multiplying by $2z$ and adding the result *with* its signs changed. That is, I multiply by $-2z$ changing the signs as I go.

Finally, adding the three partial products I have $15x^2-xy+14xz-6y^2+16yz-8z^2$, which is $5x+3y-2z$ times $3x-2y+4z$.

28. RULE.—*Multiply each term of the multiplicand by each term of the multiplier, and add the products.*

2. Multiply $3a^3b-2ab^3+b^4$ by $2ab+b^2$.

OPERATION.

$$\begin{array}{rllll}
 & 3a^3b & -2ab^3 & +b^4 & \\
 & 2ab & +b^2 & & \\
\hline
 & 6a^4b^2 & -4a^2b^4 & +2ab^5 & \\
 & & +3a^3b^3 & -2ab^5 & +b^6 \\
\hline
\textit{Prod.,} & 6a^4b^2 & +3a^3b^3 & -4a^2b^4 & +b^6
\end{array}$$

NOTE.—The pupil should give an "explanation" like the preceding. Not the "HOW" merely, but the "WHY" should be given.

3. Multiply $2a^2+4ac-c^2$ by $3a-5c$.

---

* In the literal notation it is not specially important whether we multiply from right to left, as in arithmetic, or from left to right. It is customary to proceed from left to right. Thus $5x$ times $3x$ is $15x^2$. $5x$ times $-2y$ is $-10xy$. $5x$ times $+4z$ is $+20xz$.

OPERATION.

$$\begin{array}{rl}
& 2a^2+4ac-c^2 \\
& 3a\ \ -5c \\
\hline
& 6a^3+12a^2c-3ac^2 \\
& \qquad -10a^2c-20ac^2+5c^3 \\
\hline
\textit{Prod.,} & 6a^3+\ \ 2a^2c-23ac^2+5c^3
\end{array}$$

[Pupil give the reasons.]

4. Multiply $x^2+2xy+y^2$ by $x^2-2xy+y^2$.
Prod., $x^4-2x^2y^2+y^4$.

5. Multiply $5a-3c$ by $3a$.

6. Multiply $4x-6+ax$ by $2ax^2$.

7. Multiply $12ax^2-x^3+x^2$ by $-5a^2x^2$.

8. Multiply $4b^2c+de-c$ by $3bc$.

9. Multiply $3x^2-5y+z$ by 6.

10. Multiply $6ab^2-3x^2y+7$ by -8.

11. Multiply $x+y$ by $x+y$.

12. Multiply $x-y$ by $x-y$.

13. Multiply $x-y$ by $x+y$. *Prod.*, x^2-y^2.

14. Multiply a^2+ab+b^2 by $a-b$.

15. Multiply $2b^2-2bx+x^2$ by $2b-x$.

16. Multiply b^3+b^2+b by b^3+b^2.

17. Multiply x^2-2c^2+2y by c^2+y.

18. Multiply a^2+bx+y by a^2-bx.

19. Multiply $x^4+2x^3+3x^2+2x+1$ by x^2-2x+1.

20. Multiply $2x^3-3xy+3$ by $x^2+4xy-2$.

21. Multiply $6a^3-2b^2c+3de^2$ by $a^2-5a^3b+3e^2$.

22. Multiply $1+y+y^2+y^3+y^4+y^5$ by $1-y$.
Prod., $1-y^6$.

23. Multiply $x^4+x^3y+x^2y^2+xy^3+y^4$ by $x-y$.

DIVISION

SECTION VIII.

1. How many times is 3 contained in 12? Why?*

2. How many times is a contained in $6ab$? That is, by what must a be multiplied to make $6ab$?

3. $6a^2x$ multiplied by $3ax$ is how much?

$18a^3x^2$ divided by $3ax$ is how much? Why?

4. How many times is a a factor in $3a^3b$?

What other literal factor is there in $3a^3b$?

If you take out one factor a from $3a^3b$, what is the result?

5. What is the difference between taking out a factor, and dividing by that factor?

If you take a factor 6 out of 18, what is the result?

6. If you take all the factors of 15, *i. e.*, 3 and 5, out of 75, what have you done?

* The answer is, "Because 4 times 3 make 12." And so of the following.

7. Divide 504 by 42 by taking out of 504 all the factors of 42, *i. e* , 2, 3, and 7.

8. Divide $15a^3b$ by $3a$ by taking out of $15a^3b$ the factors of $3a$.

9. If the product is + and one of the factors is +, what is the sign of the other factor? Why?

10. If the product is + and one of the factors is − what is the sign of the other factor? Why?

11. If the product is − and one of the factors is +, what is the sign of the other factor? Why?

12. If the product is − and one of the factors is −, what is the sign of the other factor? Why?

13. The product in multiplication corresponds to what in division?

The divisor corresponds to what?

Then, if the dividend is + and the divisor +, what is the sign of the quotient? Why?

If the dividend is + and the divisor −, what is the sign of the quotient?

If the dividend is − and the divisor +, what is the sign of the quotient?

If the dividend is − and the divisor −, what is the sign of the quotient?

---

29. RULE.—*To divide one monomial by another when all the factors of the divisor are found in the dividend, drop from the dividend all the factors of the divisor, and the result is the quotient. The quotient* *is + when dividend and divisor have like signs, and − when they have unlike signs.*

1. Divide $-35a^2b^3$ by $-5ab^2$. *Quot.*, $7ab$.
2. Divide $-15a^3bx^2$ by $3a^2bx$. *Quot.*, $-5ax$.
3. Divide $21a^5b^2c$ by $7a^3b^2$.
4. Divide $18b^3x^2$ by $-6bx$.
5. Divide $-36a^3b^2cx^2$ by $3ab^2cx$.
6. Divide $-9ac^2yx$ by $3ac^2x$.
7. Divide $-d^2cx^3$ by $-dcx$.
8. Divide $11b^3c^4x^7$ by b^3cx^2.
9. Divide $-13b^2c^3y^8$ by $-b^2cy^7$.

---

30. RULE.—*If there are factors in the divisor which are not in the dividend, write the divisor under the dividend in the form of a fraction, and cancel all the common factors as in fractions in arithmetic. The result is the quotient in the form of a fraction.*

1. Divide $15a^2x$ by $21acx^2$.

OPERATION. $$\frac{15a^2x}{21acx^2}=\frac{\cancel{3}\times5\cancel{a}a\cancel{x}^*}{\cancel{3}\times7\cancel{a}c\cancel{x}x}=\frac{5a}{7cx}.$$

EXPLANATION.—Since division is indicated by writing the divisor under the dividend with a line between them, we have $\frac{15a^2x}{21acx^2}$.

* This step is inserted only to exhibit the cancellation. The factors can be discerned as well without it, and in practice it should be omitted.

And since canceling like factors from numerator and denominator (as learned in arithmetic) does not change the value of a fraction, by canceling 3, a, and x, we have $\frac{5a}{7cx}$ as the quotient in the form of a fraction.

2. Divide $-17a^2bx^2$ by $-3a^3bx$. *Quot.*, $\frac{17x}{3a}$.

3. Divide $-21b^3cx$ by $17bcx^3$. *Quot.*, $-\frac{21b^2}{17x^2}$.

4. Divide $-33a^3by^2$ by $-22b^3y^3$.

5. Divide $-29b^3c^2y^2$ by $-14ab^3c^2$.

6. Divide $35a^3bx^2$ by $15ab^2x^3$.

7. Divide $27a^4bc^2$ by $-6a^2b^2c^6$.

8. Divide $15a^4x^3y$ by $-10a^2x^4y^2$.

To Divide a Polynomial by a Monomial.

1. Divide $6a^2x-10a^3x^2+4axy$ by $2ax$.

Suggestion.—If we find how many times the divisor is contained in each part (term) of the dividend and add the result, we shall find how many times it is contained in the whole.

OPERATION. $$2ax)\underline{6a^2x-10a^3x^2+4axy}$$ $$3a-5a^2x+2y \quad \textit{Quotient.}$$

31. RULE.—*Divide each term of the polynomial dividend by the monomial divisor, and write the results in connection with their own signs.*

2. Divide $2ab-6a^2x+8a^3y-2a$ by $2a$.
Quot., $b-3ax+4a^2y-1$.

3. Divide $14a^2-7ab+21ax-21a$ by $7a$.

Quot., $2a-b+3x-3$.

4. Divide $a^3x+3a^2x^2-6ax^3+3ax$ by a.

5. Divide $5b^2-10b^3+5b^4y-15b^5$ by $5b$.

6. Divide $bx^2+2x^3-8cx^4+7x^5$ by x^2.

7. Divide $4a^4-8a^3-4a^2b+8a$ by $2a$.

8. Divide $ay^4+a^2y^3-a^3y^2-ay$ by ay.

9. Divide $-y+by^2-5y^3+3y^4$ by $-y$.

Quot., $1-by+5y^2-3y^3$.

10. Divide $3b^3-9b^2+12b-15$ by $+3$.

11. Divide $-c^5+3c^4-6c^3+c^2$ by $-c^2$.

12. Divide $8y^3-4ay+a^2y^2-3y$ by y.

13. Divide $-10+20a-15a^2+20$ by -5.

14. Divide $a^3b-a^2b^2+a^3b^2-a^4b$ by ab.

15. Divide $x^4y^4+x^3y^3-x^2y^2-xy$ by xy.

To Divide one Polynomial by another.

32. A polynomial *is said to be arranged with reference to a certain letter when the term containing the highest exponent of that letter is placed first at the left, the term containing the next highest exponent next, etc.*

Thus, the polynomial $6x^2y^2+4xy^3+4x^3y+y^4+x^4$, when arranged according to the exponents of y, becomes $y^4+4xy^3+6x^2y^2+4x^3y+x^4$.

1. Arrange $2y^3x-y^2x^2+x^4+y^4$ with reference to y. Again arrange it with reference to x.

2. Arrange $2ab+a^2+b^2$ with reference to a.

3. Arrange $3a^2b^4-3a^4b^2-b^6+a^6$ with reference to a.

4. Are $x^4-a^2x^2+2a^3x-a^4$, and x^2-ax+a^2, both arranged with reference to the same letter?

What is the letter of arrangement?

If we were to *multiply* these two polynomials together in the ordinary way, would the partial products, and the entire product, be arranged with reference to any letter? What letter?

What would the first term of the product be?

What two terms would be multiplied together to make it?

Ex. 1. If $x^3-3ax^2+3a^2x-a^3$ is a product, and $x^2-2ax+a^2$ is one of the factors, what is the first term of the other factor?

By what must x^2 have been multiplied to make x^3?

2. Divide $6a^2x^2+x^4-4ax^3+a^4-4a^3x$ by x^2+a^2-2ax.

OPERATION.

DIVISOR. DIVIDEND. QUOTIENT.

$$
\begin{array}{l}
a^2-2ax+x^2)a^4-4a^3x+6a^2x^2-4ax^3+x^4(a^2-2ax+x^2\\
\qquad\qquad\quad \underline{a^4-2a^3x+\ \ a^2x^2\qquad\qquad}\\
\qquad\qquad\qquad -2a^3x+5a^2x^2-4ax^3\\
\qquad\qquad\qquad \underline{-2a^3x+4a^2x^2-2ax^3\qquad}\\
\qquad\qquad\qquad\qquad\qquad a^2x^2-2ax^3+x^4\\
\qquad\qquad\qquad\qquad\qquad \underline{a^2x^2-2ax^3+x^4}
\end{array}
$$

EXPLANATION.—Having arranged the dividend and divisor with reference to the exponents of a, and placed the divisor on the left of the dividend, as a matter of custom,* I know that the highest power of a in the dividend is produced by multiplying the highest power of a in the divisor by the highest power in the quotient. Therefore, if I divide a^4, the first term of the arranged dividend, by a^2, the first term in the arranged divisor, I get a^2 as the highest power of a in the quotient. Now, as I want to find how many times $a^2-2ax+x^2$ is contained in the dividend, and have found it contained a^2 times (and more), I can take this a^2 times the divisor out of the dividend, and then proceed to find how many times the divisor is contained in what is left of the dividend. Hence I multiply the divisor by a^2 and subtract it from the dividend, leaving $-2a^3x+5a^2x^2-4ax^3+x^4$. The same course of reasoning can be applied to this part. Thus I know that the next highest power of a in the quotient will result from dividing the first term of this remainder by the first term of the divisor, etc. When this process has terminated I have taken a^2, and $-2ax$, and x^2, times the divisor out of the dividend, and finding nothing remaining, I know that the dividend contains the divisor just $a^2-2ax+x^2$ times

33. RULE.—*Having arranged dividend and divisor with reference to the same letter, divide the first term of the dividend by the first term of the divisor for the first term of the quotient. Then subtract from the dividend the product of the divisor into this term of the quotient, and bring down as many terms to the remainder as may be necessary to form a new dividend. Divide as before, and continue the process till the work is complete.*

[NOTE.—The first three of the following examples *are* arranged for division.]

* Some prefer to put the divisor on the right, and write the quotient under it.

3. Divide $x^4-3ax^3-8a^2x^2+18a^3x-8a^4$ by $x^2+2ax-2a^2$. *Quot.*, $x^2-5ax+4a^2$.

4. Divide $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$ by $a^2-2ab+b^2$.

5. Divide $6a^4b^2+3a^3b^3-4a^2b^4+b^6$ by $3a^3b-2ab^3+b^4$.

6. Divide $4ax+4x^2+a^2$ by $2x+a$.

7. Divide $a^2b^3+b^5+a^3b^2+a^5$ by a^2-ab+b^2.

8. Divide $10ac+3c^2+3a^2+4b^2+8ab+8bc$ by $2b+a+3c$.

9. Divide $4a^5-64a$ by $2a-4$.

Operation.

$$
\begin{array}{l}
2a-4)\,4a^5-64a\,(2a^4+4a^3+8a^2+16a \\
\qquad\quad \underline{4a^5-8a^4} \\
\qquad\qquad\quad 8a^4 \\
\qquad\qquad\quad \underline{8a^4-16a^3} \\
\qquad\qquad\qquad\quad 16a^3 \\
\qquad\qquad\qquad\quad \underline{16a^3-32a^2} \\
\qquad\qquad\qquad\qquad\quad 32a^2-64a \\
\qquad\qquad\qquad\qquad\quad \underline{32a^2-64a}
\end{array}
$$

Observe that it is not necessary to bring down the term $-64a$ until we reach one in the partial subtrahends which is similar to it, *i. e.*, in this case, not until the last.

10. Divide $6y^6-96y^2$ by $3y-6$.

11. Divide a^4+4x^4 by $a^2-2ax+2x^2$.

12. Divide a^5-x^5 by $a^4+a^3x+a^2x^2+ax^3+x^4$.

13. Divide $y^4+4y^2z^2-32z^4$ by $y+2z$.

[NOTE.—If the division is not exact, the remainder may be treated as in arithmetic.]

14. Divide x^3-y^3 by $x+y$.

$$Quot.,\ x^2-xy+y^2-\frac{2y^3}{x+y}.$$

15. Divide $26x-19+2x^3-19x^2$ by $x-8$.

16. Divide x^3-y^3 by $x-y$.

17. Divide x^4-y^4 by $x+y$.

18. Divide x^4-y^4 by $x-y$.

19. Divide $4x^4-5c^2x^2+c^4$ by $2x^2-3cx+c^2$.

20. Divide $a^4-2a^2x^2+x^4$ by $a^2+2ax+x^2$.

21. Divide $12-4a-3a^2+a^3$ by $4-a^2$.

22. Divide $4y^4-9y^2+6y-1$ by $2y^2+3y-1$.

23. Divide $2x^4-32$ by $x-2$.

SECTION IX.

USE OF THE PARENTHESIS.

34.—A parenthesis ( ) *indicates that all the quantities included are to be considered as a single quantity, or to be subjected to the same operation.*

1. What is $3(a+b)$? $3a(a^2-c)$? $(a-b)(a+b)$?

What is to be done to $a+b$ in the first case?

What does the parenthesis mean which encloses a^2-c?

What is it that is to be multiplied by $3a$?
What is $a-b$ to be multiplied into?

2. Does $5ac-d$ mean the same as $5a(c-d)$?
Into what is $5a$ multiplied in the former?
Into what in the latter?

3. Do $x^2-y^2 \div x-y$ and $(x^2-y^2)\div(x-y)$ mean the same thing?
What division is indicated in the former? (Only $-y^2$ is divided by x.)
What division is indicated in $(x^2-y^2)\div(x-y)$?

4. What is $(x^3-1)\div(x-1)$?
Does $(x^3-1)\div x-1$ mean the same as the former?
What is x^3-1 to be divided by in the first case?
What in the second?

$$(x^3-1)\div x-1=x^2-\tfrac{1}{x}-1.$$
$$(x^3-1)\div(x-1)=x^2+x+1.$$

5. In the expression $3x+2y-(x-y)$ what is to be done with $x-y$?
How do you subtract a quantity?
Show that $3x+2y-(x-y)=2x+3y$.

6. Show that $x^2+2xy+y^2-(x^2-2xy+y^2)=4xy$.

7. Why is $3a-2b-(x+2)=3a-2b-x-2$?

35. When a parenthesis occurs in a polynomial, *preceded by a – sign, if the parenthesis is removed, the signs of all the terms which were within must be changed, since the sign – indicates that the quantity within the parenthesis is a subtrahend.*

8. Remove the parenthesis from $3x^2-2y-(2x-1)$.

9. If we remove the parenthesis from $3a+2b+(x-y)$ must we change the signs of the terms now within it? Why?

What is to be done with the $x-y$?

Do we change signs in order to add quantities?

10. What is the value of $4x-(2x-3)$ when $x=5$?

11. What is the value of $ab-3a\ (1-a)$ when $a=2$, and $b=3$? *Ans.*, 12.

12. What is the value of $ab+3a\ (1-a)$ when $a=2$, and $b=3$?

13. Explain that $a+(+b)=a+b$; $a+(-b)=a-b$; $a-(+b)=a-b$; $a-(-b)=a+b$.

Factoring

SECTION X.

36. The Factors *of a number are those numbers which multiplied together produce it. A* **Factor** *is, therefore, a* **Divisor.** *A* **Factor** *is also frequently called a* **Measure,** *a term arising in Geometry.*

1. What are the factors of 15? Of 21?

2. What are the factors of ab? Of $3mn$?

3. Resolve 12 into 2 factors. Resolve it into 3 factors. In how many different ways can 12 be resolved into 2 factors?

4. Is $3a$ a factor of $3ax^3$?

What is the other factor?

Why are $3a$ and x^2 the factors of $3ax^2$?

Are $3ax$ and x also the factors of $3ax^2$?

Are $3x^2$ and a also the factors of $3ax^2$?

Are 3 and ax^2? Are $3x$ and ax?

5. Resolve $15a^2x^3$ into two factors in as many different ways as you can.

Resolve $15a^2x^3$ into as many different factors as you can.*

87. A Monomial *can be resolved into as many different sets of factors as there can be made groups of the factors of the numerical co-efficient, and the literal factors which enter into it.*

---

1. Four times x and 3 times x are $(4+3)$ times x, or $(4+3)\,x$.

In like manner how many times x are a times x and b times x?

2. $2a^2$ times y and $3b$ times y are how many times y? How is the result written? *Ans.*, $(2a^2+3b)y$.

3. Write by use of the parenthesis $3m$ times x, plus $2n$ times x, plus x.

4. Five times x minus 3 times x are $(5-3)$ times x, or $(5-3)\,x$.

In like manner write $2a$ times x minus $3b$ times x.

* Of course this means without the use of fractional exponents of which the pupil is supposed to know nothing as yet.

5. Write 3 times y, minus $2b$ times y, plus y, by use of the parenthesis.

6. If we consider $5a$, $2b$, and c, as the co-efficients of x in the terms $5ax$, $2bx$, and cx, how many times x have we? How is it written?

7. Considering $3m^2$, 5, and $6ab^2$ as the co-efficients of y, what is the sum of $3m^2y$, $5y$, and $6ab^2y$?

Ans., $(3m^2+5+6ab^2)y$.

38. This process *is called adding the terms with reference to the common letter* **y.**

8. Add with reference to x, the monomials $2x$, $3nx$, and abx.

9. Add with reference to y, the monomials $2ay$, $-3by$, y, $-cy$ and ay. *Sum*, $(3a-3b+1-c)y$.

10. Add with reference to x, a^2x, and $-x$.

11. Add with reference to y, $2ay$, and $-3by$.

Removing a Letter which is a Common Factor, from each Term of a Polynomial.

1. What is the product of a and $a-x$? What are the factors of a^2-ax?

2. What is the product of 3 and $a+b$? What are the factors of $3a+3b$?

3. What is the product of $3a^2$ and $1-a$? What are the factors of $3a^2-3a^3$?

4. What is the product of $2a-2b$ and x? What are the factors of $2ax-2bx$?

39. Any factor *which occurs in every term of a polynomial can be removed by dividing each term of the polynomial by it.*

5. What are the factors of a^2x-x?

6. Show that $mx^2+nx^2=(m+n)\,x^2$?

7. Write $2ax-x$ as the product of a binomial factor into x.

8. Write $12ax^2-3x^2+x^2$ as the product of a trinomial factor into x^2.

The Square of the Sum of Two Numbers.

1. What is the product of $a+b$ and $a+b$?
What are the factors of $a^2+2ab+b^2$?

2. What is the product of $x+y$ and $x+y$?
What are the factors of $x^2+2xy+y^2$?

3. What is the product of $1+x$ and $1+x$?
What are the factors of $1+2x+x^2$?

40. We see *from the last examples that the square of the sum of two numbers equals the square of one of them, + twice the product of the two, + the square of the other. Thus* $(a+b)(a+b)$ *is the square of the sum of the two numbers* a *and* b, *and is equal to* $a^2+2ab+b^2$.

4. Resolve $a^2+4ax+4x^2$ into its factors.

5. Resolve $4a^2+12ax+9x^2$ into its factors.

6. Resolve $1+4xy+4x^2y^2$ into its factors.

The Square of the Difference of Two Numbers.

1. What is the product of $x-y$ and $x-y$?
What are the factors of $x^2-2xy+y^2$?

2. What is the product of $m-n$ and $m-n$?
What are the factors of $m^2-2mn+n^2$?

3. What is the product of $1-x$ and $1-x$?
What are the factors of $1-2x+x^2$?

4. What is the product of $2-x$ and $2-x$?
What are the factors of $4-4x+x^2$?

41. From these examples, *we see that the square of the difference of two numbers is equal to the square of one of them, − twice the product of the two, + the square of the other. Thus* $(x-y)(x-y)=x^2-2xy+y^2$.

5. Resolve $9a^2-6ay+y^2$ into its factors.

6. Resolve $a^2x^2-2ax+1$ into its factors.

7. Resolve $4a^2x^2-12abxy+9b^2y^2$ into its factors.

The Product of the Sum and Difference of Two Numbers.

1. What is the product of $x+y$ and $x-y$?
What are the factors of x^2-y^2?

2. What is the product of $a+b$ and $a-b$?
What are the factors of a^2-b^2?

3. What is the product of $1+x$ and $1-x$?
What are the factors of $1-x^2$?

4. What is the product of $2+x$ and $2-x$?
What are the factors of $4-x^2$?

42. We see, from these examples, *that the product of the sum and difference of two numbers is equal to the difference of their squares.* *Thus* $(x+y)(x-y)=x^2-y^2$.

5. Resolve $4x^2-9y^2$ into its factors.

6. Resolve $a^2x^2-c^2y^2$ into its factors.

7. Resolve m^2n^2-16 into its factors.

Divisors of the Difference or Sum of the same Powers of Two Numbers.

1. Will $x-y$ divide x^4-y^4 without remainder? Try it.
Will it divide x^3-y^3?

2. Will $x-y$ divide x^6-y^6 without remainder? Try it.
Will it divide x^5-y^5?

3. Will $x+y$ divide x^4-y^4?
Will it divide x^4+y^4?
Will it divide x^5+y^5?
Try it in each case.

43. The difference *between any two quantities is a divisor of the* **difference** *between the same powers of the quantities.*

The **sum** *of two quantities is a divisor of the* **difference** *of the same* **even** *powers, and the* **sum** *of the same* **odd** *powers of the quantities.**

4. Is $a-b$ a factor of a^3-b^3? If so, what is the other factor?

5. Is $a+b$ a factor of a^3+b^3? If so, what is the other factor?

COMMON DIVISORS

SECTION XI.

1. Will 3 divide both 12 and 15 without a remainder?
Is 3 therefore a divisor of 12?
Is it also a divisor of 15?
Is it a *Common Divisor* of both 12 and 15?

2. Will a divide both $3ax$ and $2a^2$ without a remainder?
Is a therefore a divisor of $3ax$?
Is it also a divisor of $2a^2$?
Is it a *Common Divisor* of both $3ax$ and $2a^2$?

44. A Common Divisor *is a common integral*† *factor of two or more numbers.*

---

* It is not deemed expedient to give demonstrations of these propositions. Let the pupil learn them, and how to apply them.

† Since the genius of the literal notation makes the letters strictly

3. What common divisor have $3ay$ and $2a^2x$?
Have they more than one common divisor?

4. What common divisor have $15a^2xy$ and $12abx^2$?
Have they more than one common divisor?
How many?

5. What common divisors have $14a^3x^2$ and $21a^2cx^3$?
Is $7ax$ a common divisor?
Is $7a^2x$? Is ac? Is $7c$? Is $7a^2x^2$?
What is the common divisor of $14a^3x^2$ and $21a^2cx^3$ which contains the *highest* number of factors?

6. What are *all* the common factors, or divisors, of $6a^3by^6$ and $4a^2cy^4$?
What is the product of *all* the common factors of these numbers?
What then is their common divisor which has the highest number of factors?

45. The Highest Common Divisor *of two numbers is the product of all their common prime factors.*

7. What is the H. C. D.* of $10a^3y^5$ and $4a^2by^3$?

8. What is the H. C. D. of $12mn^2x^2y$ and $30m^2n^3y$?

46. A Prime Number *is one which has no integral factor other than itself and unity.*

general in their signification, we cannot affirm that a, or b, or x, is an integral factor. We can only affirm with strict propriety that such have the *integral form*, a may be a fraction or a mixed number; as also may any letter. But this discrimination need not be urged upon quite young pupils, however important it is in itself, since failure to notice it will give them no *practical* difficulty.

* H. C. D means *highest common divisor.*

47. **RULE.**—*To find the H. C. D. of two numbers, resolve them into their prime factors, and take the product of all the common factors.*

Highest Common Divisor of Polynomials.

1. What are the prime factors of $2ax^2-2ay^2$, and $6x^2+12xy+6y^2$? What the H. C. D.?

Suggestion. $2ax^2-2ay^2=2a(x^2-y^2)=2a(x-y)(x+y)$. $6x^2+12xy+6y^2=6(x^2+2xy+y^2)=2\cdot3(x+y)(x+y)$. We now see that 2 and $x+y$ are the only common factors. Hence their product, $2x+2y$, is the H. C. D.

2. What is the H. C. D. of $3a^2x+4a^3x$, and $a^2y-5a^2y^3$?

3. What is the H. C. D. of $5a^3c+10a^3c^2$, and $10a^2b-5a^3bx$?

4. What is the H. C. D. of $2ax^2-6a^2y$, and $4a^2x+8ay^2-2a$? *Ans.*, $2a$.

5. What is the H. C. D. of $ax^2+2a^2x+a^3$, and ax^3-a^3x?

6. What is the H. C. D. of $3ab^2-12ax^2$, and $6a^2b^2+24a^2bx+24a^2x^2$?

7. What is the H. C. D. of $4bx^3-4by^3$, and $8b^2x^4-8b^2x^2y^2$? *Ans.*, $4bx-4by$,

8. What is the H. C. D. of $10ax+3ay$, and $14x^2y+21axy^2$?

COMMON MULTIPLES.

SECTION XII.

48. A Common Multiple *of two or more numbers is an integral number which contains each of them as a factor, or which is divisible by each of them.*

1. What are all the factors of 30?
Will any number contain 30 which does not contain all the factors of 30?

2. What are the factors of 6?
Will any number contain 6 which does not contain the factors 2 and 3?
Does 9 contain both of the factors of 6?
Will 9 contain 6?
What factor is lacking?

3. Will 56 contain 15?
What factors of 15 does 56 lack?
Does 54 contain all the factors of 15?
Does it contain 15?
What factor of 15 does it lack?

4. Does 42 contain all the factors of 6?
Does it contain any other factors than those of 6?
Does its containing other factors prevent its containing 6?
Is 42 a multiple of 6? Why?

5. Does $12a^2x$ contain all the factors of $3ax$?
Does it contain any other factors than those of $3ax$?
Is $12a^2x$ a multiple of $3ax$?

6. Is $16m^2y^2$ a multiple of $8mx$? Why not?
Does it contain all the factors of $8mx$?
What one is lacking?
If you put in the lacking factor, will it ($16m^2xy^2$) be a multiple of $8mx$?

7. What factors must you put into $3a^2y^3$ to make it a multiple of $6a^3y$?

49. One number, *to be a multiple of another, must contain* **all** *its factors.*

8. Is 16 a multiple of both 8 and 4?
Does 16 contain all the factors of 8?
All the factors of 4?

9. Is $42a^2x^3$ a common multiple of $6ax$ and $7ax^2$, *i. e.*, does it contain all the factors of each?

10. Can a multiple of any number be less, or of lower degree, than the number itself?

Least or Lowest Multiples.

50. A number *is its own least or lowest multiple.*

1. Can 70 and 42 have a common multiple less than 70?
Can any two numbers have a common multiple less than the greater of the two?
Is 70 a common multiple of 70 and 42? Why not?
Ans. Because it does not contain 42; and it does not contain 42 because it does not contain all the factors of 42.

What factor must be put in with 70 in order to make a product which will contain 42?

51. Two numbers *cannot have a common multiple less, or of lower degree, than the larger, or the one of the higher degree.*

2. Can $5a^3b^2$ and $7a^2bx$ have a common multiple which does not contain all the factors of $5a^3b^2$?

Can they have one which does not contain all the factors of $7a^2bx$? Why?

What factors has $7a^2bx$ which $5a^3b^2$ has not?

If we multiply $5a^3b^2$ by all the factors of $7a^2bx$, which it does not already contain, will the product be a common multiple of the two numbers?

52. The Least Common Multiple *of two or more numbers is the least integral number which is divisible by each of them.*

53. RULE.—*To find the L. C. M. of two or more literal numbers, multiply the number of highest degree by all the factors of the next highest number which are not found in it; this product by all the factors in the next highest which are not found in the preceding, and so proceed with all the numbers. This product is the L. C. M.*

REASON.—This product contains each number because it contains all the factors of each. It is the L. C. M. because it contains no unnecessary factors.

3. What is the L. C. M. of $x^2+2xy+y^2$ and $3ax^2-3ay^2$?

Suggestion.—The factors of $x^2+2xy+y^2$ are $x+y$ and $x+y$. Of $3ax^2-3ay^2$ the factors are $3a$, $x-y$, and $x+y$. Hence $(3ax^2-3ay^2)(x+y)$ is the L. C. M.

* L. C. M. signifies *least common multiple.*

4. Find the lowest common multiple of ax^2, $2a^2y$, $4y+y^2$, and ax^2+4x^2.

L. C. M., $8a^3x^2y+2a^3x^2y^2+32a^2x^2y+8a^2x^2y^2$.

5. Find the lowest common multiple of $2ay^2$, $4ay^2$, $2x-4x^2$, and x^3+ax^2.

L. C. M., $4ax^3y^2-8ax^4y^2+4a^2x^2y^2-8a^2x^3y^2$.

6. Find the lowest common multiple of $3a^2$, ax^2, $3a+6a^2$, and x^3-3x^2.

L. C. M., $3a^2x^3+6a^3x^3-9a^2x^2-18a^3x^2$.

7. Find the lowest common multiple of $4y^3$, $2ay^2$, $5a-5ab$, and $10a-5$.

8. Find the lowest common multiple of $5a$, $10ab$, $3y+3y^2$, and $6y^3+3y^2$.

FRACTIONS

SECTION XIII.

54. For the various operations *upon fractions in the* **literal notation,** *the ordinary rules of arithmetic for the corresponding cases apply, only that the fundamental operations of addition, subtraction, multiplication and division, are performed by the preceding rules.*

55. It is to be observed, *however, that the conception to be taken of a fraction, in the literal notation, is, that it is an indicated problem in division, the numerator corresponding to the dividend, and the denominator to the divisor; and hence, that the* **value**

*of the fraction is the quotient of the numerator divided by the denominator. All the operations are to be explained on these principles.**

56. The same terms, *as Integer, Entire, Mixed, Proper, Improper, etc., are used in reference to fractions in the literal notation as are used in arithmetic, and with a significance so similar that the definitions need not be repeated.*

57. Reduction, *in mathematics, is changing the form of an expression without changing its value.*

Reduction to Lowest Terms.

58. RULE.—*To reduce fractions to their lowest terms, reject all common factors from both terms, or divide both terms by their H. C. D.*

REASON.—The numerator being the dividend and the denominator the divisor, rejecting common factors does not alter the quotient, which is the value of the fraction. Thus it is evident that if a divisor is contained in a dividend 8 times, half that divisor will be contained in half that dividend 8 times, etc. Hence, rejecting common factors from numerator and denominator does not alter the value of the fraction. Again, when all common factors are rejected, the fraction has the lowest possible numerator and denominator, since to reject any factor from either which is not common would change the value of the fraction.

---

* It will not do to say of $\frac{a}{b}$, that b indicates that a quantity is to be separated into b equal parts, etc., since the genius of the literal notation requires that b represent *any* quantity, fractional as well as integral.

1. Reduce $\frac{25a^2m^2x}{15am^3x^2}$ to its lowest terms.

OPERATION. $$\frac{25a^2m^2x}{15am^3x^2}=\frac{5\cdot\cancel{5}\cancel{a}a\cancel{m}\cancel{m}\cancel{x}^*}{3\cdot\cancel{5}\cancel{a}m\cancel{m}\cancel{m}\cancel{x}x}=\frac{5a}{3mx}.$$

2. Reduce $\frac{105b^2y^3}{15by^2}$ to its lowest terms.

3. Reduce $\frac{12a^2b^2-16a^3b}{8a^2bx+4a^5b^2}$ to its lowest terms.

Suggestion.—Divide numerator and denominator by $4a^2b$.

4. Reduce $\frac{28x^2y^2-42x^3y^2}{7x^3y}$ to its lowest terms.

5. Reduce $\frac{14a^2x^2-21ax^3+56a^3x^2}{28a^3x^3-35ax^4}$ to its lowest terms.

Suggestion.—In attempting to reduce such a fraction, first observe whether there is a common factor in the numerical co-efficients. In this case 7 is such a factor. Then notice if any letter or letters are common to all the terms, and the term or terms which contain the *lowest* powers of these letters. In this case a and x are in each term. There are two terms which have simply a, and two which have x^2. Hence, we can divide numerator and denominator of the fraction by $7ax^2$. Could we by $7a^2x^2$? By $7ax^3$? Why not? After having removed the monomial factors, examine the resulting fraction and see if any binomial factor exists. In this case we have $\frac{2a-3x+8a^2}{4a^2x-5x^2}$. If there is a binomial factor, we can

---

* Of course in practice this step is omitted. But the student should see that it is always implied, and it is a convenient form to indicate the cancellation.

detect it easiest in the denominator, since, if this can be factored it must be by (42).

6. Reduce $\dfrac{12a^4x^2y^2-18a^3x^3y^2+42a^3x^4y^2}{6a^3x^2y^2-6a^5x^3y^2}$ to its lowest terms. *Result*, $\dfrac{2a-3x+7x^2}{1-a^2x}$.

7. Reduce $\dfrac{4a^2x+8a^2x^3-12a^3x^2}{4a^2x^3-20a^3x+28a^2xy}$ to its lowest terms.

8. Reduce $\dfrac{2bx^3-2b^3x}{2bx^2+4b^2x+2b^3}$ to its lowest terms.

Suggestion.—Having removed the monomial factor there results $\dfrac{x^3-b^2x}{x^2+2bx+b^2}$. Factoring the numerator it becomes $x(x^2-b^2)$ or $x(x-b)(x+b)$, by (42). Now the question is whether any factor of the numerator is found in the denominator, and hence can be canceled.

9. Reduce $\dfrac{ax+x^2}{ab^2+b^2x}$ to its lowest terms.

10. Reduce $\dfrac{x^3+y^3}{x^2-y^2}$ to its lowest terms. *Result*, $\dfrac{x^2-xy+y^2}{x-y}$.

11. Reduce $\dfrac{a^6-a^2x^4}{3a^6+3a^5x-3a^4x^2-3a^3x^3}$ to its lowest terms.

12. Reduce $\dfrac{x^2+2xy+y^2}{x^3-xy^2}$ to its lowest terms.

To Reduce a Fraction from an Improper to an Integral or Mixed form.

59. RULE.—*Perform the division indicated.*

REASON.—As a fraction is but an *indicated* operation in division we may *perform* the operation and get the quotient, which is the value of the fraction.

1. Reduce $\frac{x^2-2ax+a^2}{x-a}$ to an integral form.

Result, $x-a$.

Suggestion.—Divide the numerator by the denominator.

2. Reduce $\frac{10a^2-13ax-3x^2}{2a-3x}$ to an integral form.

3. Reduce $\frac{12c^3+8ac^2x^2-3acx-2a^2x^3}{4c^2-ax}$ to an integral form.

4. Reduce $\frac{10a^2-13ax-3x^2}{5a+x}$ to an integral form.

5. Reduce $\frac{12c^3+8ac^2x^2-3acx-2a^2x^3}{3c+2ax^2}$ to an integral form.

6. Reduce $\frac{ay+b}{y}$ to an integral or mixed form.

OPERATION. $y)\underline{ay+b}.$ $a+\frac{b}{y}$ The term ay can be divided by y, giving a. But we can only indicate the division of b by y, by

writing it in the form $\frac{b}{y}$; and as the sign of both b and y is $+$, the quotient, $\frac{b}{y}$, is $+$, and is to be added to a.

7. Reduce $\frac{20x^3-10x+4}{5x}$ to an integral or mixed form.

Result, $4x^2-2+\frac{4}{5x}$.

8. Reduce $\frac{2a^2x-x^3}{a}$ to an integral or mixed form.

Result, $2ax-\frac{x^3}{a}$.

Query.—Why has $\frac{x^3}{a}$ the $-$ sign?

9. Reduce $\frac{a^3+x^3-x^4}{a+x}$ to an integral or mixed form.

Result, $a^2-ax+x^2-\frac{x^4}{a+x}$.

10. Reduce $\frac{3ax+b^2}{x}$ to an integral or mixed form.

11. Reduce $\frac{4a^2+2ab-b^2}{2a+3b}$ to a mixed form.

12. Reduce $\frac{a^3-b^3}{a+b}$ to a mixed form.

13. Reduce $\frac{6a^2+5ax-x^2}{3a^2+2ax}$.

14. Reduce $\frac{27a^3-3b^2-4x+9a^2}{9a^2}$.

To Reduce Mixed forms to Improper forms.

60. RULE.—*Multiply the integral part by the given denominator, and annexing the numerator of the fractional part, if any, write the sum over the given denominator.*

DEM.*—In the case of a number in the integral form, the process consists of multiplying the given number by the given denominator and indicating the division of the product by the same number, and hence is equivalent to multiplying and dividing by the same quantity, which does not change the value of the number. The same is true as far as relates to the integral part of a mixed form, after which the two fractional parts are to be added together. As they have the same divisors, the dividends can be added upon the principle that the sum of the quotients equals the quotient of the sum.

1. Reduce $2a-x^2+\frac{3ax-4a^2}{a-x}$ to a fractional form.

MODEL SOLUTION.†—Multiplying $2a-x^2$ by $a-x$, I have $2a^2-ax^2-2ax+x^3$, which divided by $a-x$, of course equals $2a-x^2$; or $2a-x^2=\frac{2a^2-ax^2-2ax+x^3}{a-x}$. $\therefore 2a-x^2+\frac{3ax-4a^2}{a-x}=\frac{2a^2-ax^2-2ax+x^3}{a-x}+\frac{3ax-4a^2}{a-x}$. But, as the sum of these two

* This abbreviation stands for "*Demonstration.*" A demonstration is a proof of the correctness of a rule, or the truth of a statement. It is a more formal and scientific statement of what has heretofore been called "*Reasons.*"

† These solutions are designed as models for the pupil in the explanations given by him from the blackboard. In the earlier part of this very elementary treatise it is thought that the pupil is introduced to so many new ideas, that it would be unwise to attempt much rigor in regard to his *style* of explanation.

quotients equals the quotient of the sum, I have, after uniting similar terms, $\frac{x^3-ax^2+ax-2a^2}{a-x}$.

2. Reduce $2x-\frac{3y}{x}$ to the form of a fraction.

3. Reduce $a+x-\frac{a^2+2ax+x^2}{a-x}$ to the form of a fraction.

Suggestions.—The integral part, $a+x$, multiplied by the denominator, $a-x$, gives a^2-x^2. Now notice that the numerator, $a^2+2ax+x^2$, is to be *subtracted* from this, as the sign before the fraction is $-$. If we subtract $a^2+2ax+x^2$ from a^2-x^2, we have $-2ax-2x^2$. Hence the result is $\frac{-2ax-2x^2}{a-x}$, or $-\frac{2x(a+x)}{a-x}$.

We may also write this thus:

$a+x-\frac{a^2+2ax+x^2}{a-x}=\frac{a^2-x^2-(a^2+2ax+x^2)}{a-x}$. Now the quantity in the parenthesis is to be subtracted from the a^2-x^2. Hence, changing the signs of the terms in the parenthesis, and dropping the marks, we have $\frac{a^2-x^2-a^2-2ax-x^2}{a-x}$. Uniting similar terms, this becomes $\frac{-2ax-2x^2}{a-x}$.

4. Reduce $1-\frac{a^2-2ab+b^2}{a^2+b^2}$ to the form of a fraction. *Result,* $\frac{2ab}{a^2+b^2}$.

5. Reduce $a-x+\frac{a^2+x^2-5}{a+x}$ to the form of a fraction. *Result,* $\frac{2a^2-5}{a+x}$.

6. Reduce $3x-\frac{4x^2-5}{5x}$ to the form of a fraction.

7. Reduce $4ax-\frac{3ab}{y}$ to the form of a fraction.

8. Reduce $7x-3y+\frac{x^2+y^2}{x-y}$ to the form of a fraction.

9. Reduce $a+x-\frac{a^2+x^2}{a-x}$ to the form of a fraction.

10. Reduce $a^2-ax+x^2-\frac{x^3}{a+x}$ to the form of a fraction.

11. Reduce $x+y-\frac{x^2-y^2+7}{x-y}$ to the form of a fraction.

To Reduce Fractions having different Denominators to Equivalent Fractions having a Common Denominator.

61. RULE.—*Multiply both terms of each fraction by the denominators of all the other fractions.*

DEM.—This gives a common denominator, because each denominator is the product of all the denominators of the several fractions. The value of any one of the fractions is not changed, because both numerator and denominator are multiplied by the same number.

1. Reduce the fractions $\frac{x}{y}$, $\frac{2-b}{a+b}$, and $\frac{3a-x}{a-b}$, to equivalent fractions having a common denominator.

MODEL SOLUTION.—Multiplying both terms of the fraction $\frac{x}{y}$ by

$a+b$ and $a-b$, or by a^2-b^2, I have $\frac{a^2x-b^2x}{a^2y-b^2y}$, which has the same value as $\frac{x}{y}$, since the numerator and denominator have been multiplied by the same number. In like manner multiplying both terms of $\frac{2-b}{a+b}$ by y and $a-b$, I have $\frac{2ay-aby-2by+b^2y}{a^2y-b^2y}$, the value of which is the same as $\frac{2-b}{a+b}$, since, etc. Finally, multiplying both terms of $\frac{3a-x}{a-b}$, by y and $a+b$, I have $\frac{3a^2y-axy+3aby-bxy}{a^2y-b^2y}$, which has the same value as $\frac{3a-x}{a-b}$, since, etc. These fractions have the common denominator a^2y-b^2y, as in each case the new denominator is the product of all the old ones.

2. Reduce $\frac{a}{x}$, $\frac{b}{y}$, and $\frac{c}{z}$, to equivalent fractions having a common denominator.

3. Reduce $\frac{4}{by}$, $\frac{2b}{y}$, and $\frac{3}{4}$, to equivalent fractions having a common denominator.

4 Reduce $\frac{a}{x+y}$, $\frac{b}{x-y}$, and $\frac{c}{3}$, to equivalent fractions having a common denominator.

To Reduce Fractions to Equivalent Fractions having the Lowest Common Denominator.

62. RULE.—*Find the lowest common multiple of all the denominators, for the new denominator. Then multiply both numerator and denominator of each* *fraction by the quotient of the lowest common denominator divided by the denominator of that fraction.*

DEM.—The purpose in getting the L. C. M. is to get the *lowest* number which can be divided by each of the denominators. The process does not change the value of the fractions, since it does not alter the quotient to multiply dividend and divisor by the same number.

5. Reduce $\frac{a}{1-a}$, $\frac{a^2}{(1-a)^2}$, and $\frac{a^3}{(1-a)^3}$, to equivalent fractions having the lowest common denominator.

MODEL SOLUTION.

OPERATION.—The L. C. M. of $1-a$, $(1-a)^2$, and $(1-a)^3$ is $(1-a)^3$. $\frac{a\times(1-a)^2}{(1-a)\times(1-a)^2}=\frac{a-2a^2+a^3}{1-3a+3a^2-a^3}$, $\frac{a^2\times(1-a)}{(1-a)^2\times(1-a)}=\frac{a^2-a^3}{1-3a+3a^2-a^3}$, and $\frac{a^3}{(1-a)^3}=\frac{a^3}{1-3a+3a^2-a^3}$.

EXPLANATION.—By inspection I observe that $(1-a)^3$ is the L. C. M. of the denominators, since it is the lowest number which contains itself, and it also contains each of the other denominators. Now, to make the denominator of $\frac{a}{1-a}$, $(1-a)^3$, I must multiply it by $(1-a)^3 \div (1-a)$; *i. e.*, by $(1-a)^2$. But to preserve the value of the fraction, I must multiply the numerator by the same quantity. Thus $\frac{a}{1-a}=\frac{a(1-a)^2}{(1-a)^3}=\frac{a-2a^2+a^3}{1-3a+3a^2-a^3}$; etc.

6. Reduce $\frac{x^2}{4}$, $\frac{a}{3y}$, and $\frac{a-x}{6y^2}$, to equivalent fractions having the lowest common denominator.

Equivalent fractions $\frac{3x^2y^2}{12y^2}$, $\frac{4ay}{12y^2}$, $\frac{2a-2x}{12y^2}$.

7. Reduce $\frac{a^2}{a^2-x^2}$, $\frac{1}{a-x}$, and $\frac{a}{a+x}$, to equivalent fractions having the lowest common denominator.

8 to 11. Reduce the following sets of fractions to equivalent sets having the lowest common denominators.

8. $\frac{a}{x}$, $\frac{b}{x^2}$, and $\frac{1}{x^3}$.

9. $\frac{y}{4}$, $\frac{a}{2x^2}$, and $\frac{a^2}{2+x}$.

10. $\frac{a^2}{a^2-x^2}$, $\frac{a}{a+x}$, and $\frac{1}{a-x}$.

11. $\frac{3}{2x^3-2xy^2}$, and $\frac{5}{6(x+y)^2}$.

To Add Fractions.

1. Twelve divided by 3, +15 divided by 3, +21 divided by 3, makes how much divided by 3?

2. Eight divided by 5, +11 divided by 5, +3 divided by 5, makes how much divided by 5?

3. a divided by b, $+c$ divided by b, $+d$ divided by b, makes how much divided by b?

4. Is $\frac{m}{n}+\frac{x}{n}+\frac{y}{n}$ equal to $\frac{m+x+y}{n}$? Why?

63. The sum of the quotients *of several numbers divided by the same number is equal to the sum of the numbers divided by this common divisor.*

64. RULE.—*To add fractions reduce them to forms having a common denominator, if they have not such a form, and then add the numerators, and write the sum over the common denominator.*

DEM.—The reduction of the several fractions to forms having a common denominator, if they have not one, does not alter their values (62), and hence does not alter the sum. Then, when they have a common denominator (divisor), the sum of the several quotients is equal to the quotient of the sum of the several dividends divided by the common divisor, or denominator (63).

5. Add $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{5}$. *Sum,* $\frac{31x}{30}$.

MODEL SOLUTION.—Reducing the fractions to equivalent ones having a common denominator, I have

$$\frac{x}{2}=\frac{3\cdot5x}{3\cdot5\cdot2}=\frac{15x}{30},$$

$$\frac{x}{3}=\frac{2\cdot5x}{2\cdot5\cdot3}=\frac{10x}{30},$$

$$\frac{x}{5}=\frac{2\cdot3x}{2\cdot3\cdot5}=\frac{6x}{30}.$$

As the new fractions are equivalent to the given ones, I have $\frac{x}{2}+\frac{x}{3}+\frac{x}{5}=\frac{15x}{30}+\frac{10x}{30}+\frac{6x}{30}=\frac{31x}{30}$, since the sum of the quotients is equal to the quotient of the sum of several numbers divided by the same number (63).

6. Add $\frac{2x}{3}$, and $\frac{x+2}{2}$. *Sum,* $\frac{7x+6}{6}$.

7. Add $\frac{2-x}{5}$, and $\frac{3x-1}{2}$. *Sum*, $\frac{13x-1}{10}$.

8. Add $\frac{a+b}{a-b}$, and $\frac{a-b}{a+b}$. *Sum*, $\frac{2a^2+2b^2}{a^2-b^2}$.

9. Add $\frac{1-x^2}{1+x^2}$, and $\frac{1+x^2}{1-x^2}$. *Sum*, $\frac{2+2x^4}{1-x^4}$.

10. Add $\frac{1}{1+x}$, and $\frac{1}{1-x^2}$.

11. Add $\frac{x}{x-1}$, and $\frac{x}{x+2}$.

12. Add $\frac{1+x^2}{1-x^2}$, and $\frac{1-x^2}{1+x^2}$.

13. Add $\frac{a^2}{a^2-x^2}$, $\frac{1}{a-x}$, and $-\frac{a}{a+x}$.

Sum, $\frac{ax+a+x}{a^2-x^2}$.

14. Add $3x$, $x+\frac{3a}{4}$, and $4x-\frac{6a}{5}$.

Suggestion.—In adding mixed numbers it is usually best to add the integral and fractional parts separately.

15. Add $4x$, $3x+\frac{2x-5}{3}$, and $5x-\frac{x-1}{2x}$.

16. Add $\frac{1}{x^2-y^2}$ and $\frac{2xy}{x^4-y^4}$.

Sum, $\frac{x+y}{x^3-x^2y+xy^2-y^3}$.

To Subtract Fractions.

1. Ten divided by 2, minus 4 divided by 2, make how much divided by 2?

2. Seventeen divided by 3, minus 11 divided by 3, make how much divided by 3?

3. a divided by b, minus c divided by b, make how much divided by b?

65. The difference *between the quotients of two numbers divided by the same number, is equal to the difference between the two numbers divided by this common divisor.*

66. RULE.—*To subtract fractions reduce the fractions to forms having a common denominator, if they have not that form, and subtract the numerator of the subtrahend from the numerator of the minuend, and place the remainder over the common denominator.*

DEM.—The value of the fractions not being altered by reducing them to a common denominator, their difference is not altered. After this reduction, we have the difference of two quotients arising from dividing two numbers (the numerators) by the same divisor (the common denominator). But this is the same as the quotient arising from dividing the difference between the numbers by the common divisor (65).

4. From $\frac{x+y}{x-y}$ subtract $\frac{x-y}{x+y}$.

MODEL SOLUTION.

OPERATION. $(x-y)(x+y) = x^2-y^2$,

$$\frac{(x+y)\times(x+y)}{(x-y)\times(x+y)} = \frac{x^2+2xy+y^2}{x^2-y^2},$$

$$\frac{(x-y)\times(x-y)}{(x+y)\times(x-y)} = \frac{x^2-2xy+y^2}{x^2-y^2},$$

$$(x^2+2xy+y^2)-(x^2-2xy+y^2) = 4xy.$$

$$\therefore \frac{x+y}{x-y} - \frac{x-y}{x+y} = \frac{4xy}{x^2-y^2}.$$

EXPLANATION.—The L. C. M. of $x-y$ and $x+y$ is their product, since they have no common factor. Hence x^2-y^2 is the L. C. D. To reduce $\frac{x+y}{x-y}$ to this denominator I multiply both its terms by $x+y$, which gives $\frac{x^2+2xy+x^2}{x^2-y^2}$. In like manner multiplying both terms of $\frac{x-y}{x+y}$ by $x-y$, I have $\frac{x^2-2xy+y^2}{x^2-y^2}$. I have now to subtract $\frac{x^2-2xy+y^2}{x^2-y^2}$ from $\frac{x^2+2xy+y^2}{x^2-y^2}$. Since the difference of the quotients of two numbers divided by the same number is the same as the quotient arising from dividing the difference between those numbers by the common divisor, I take the difference of the numerators (the quantities to be divided) which is $4xy$, and dividing it by x^2-y^2 I have $\frac{4xy}{x^2-y^2}$, for the remainder of $\frac{x+y}{x-y}$ less $\frac{x-y}{x+y}$.

5. From $\frac{2x}{3}$ take $\frac{x}{4}$. *Rem.*, $\frac{5x}{12}$.

6. From $\frac{a}{3}$ take $\frac{b}{2}$.

7. From $\frac{x-1}{3}$ take $\frac{x+2}{5}$.

8. From $\frac{1}{x-y}$ take $\frac{1}{x+y}$.

9. From $\frac{1-x^2}{1+x^2}$ take $\frac{1+x^2}{1-x^2}$. *Rem.*, $\frac{-4x^2}{1-x^4}$.

10. From $\dfrac{a^2}{1+x}$ subtract $\dfrac{b}{1-x^2}$.

11. From $\dfrac{1}{x-y}$ subtract $\dfrac{1}{x^2-y^2}$.

12. From $2a-3x+\dfrac{a-x}{a}$ subtract $a-5x+\dfrac{x-a}{x}$.

Suggestion.—In accordance with the use of the parenthesis (34), we may *indicate* the subtraction thus:

$$2a-3x+\frac{a-x}{a}-\left(a-5x+\frac{x-a}{x}\right).$$

Now removing the parenthesis by (35) we have

$$2a-3x+\frac{a-x}{a}-a+5x-\frac{x-a}{x}.$$

Uniting the similar terms and adding the fractions, we have $a+2x+\dfrac{a^2-x^2}{ax}$.

13. From $a+x+\dfrac{x}{x^2-y^2}$ take $a-x+\dfrac{1}{x+y}$.

14. From $5x+\dfrac{4x-6}{7}$ take $2x+\dfrac{7x-12}{13}$.

15. From $6a+\dfrac{3x-2a}{a}$ take $2a+\dfrac{4a-3x}{x}$.

16. From $\dfrac{2y^2-2y+1}{y^2-y}$ take $\dfrac{y}{y-1}$. *Rem.*, $1-\dfrac{1}{y}$.

Multiplication and Division of Fractions.

67. There are two cases: *1st, when the multiplier or divisor is an integer; and 2d, when the multiplier or divisor is a fraction.*

Case I. In Multiplication.

68. RULE.—*To multiply a fraction by an integer multiply the numerator or divide the denominator.*

DEM.—The reason for this rule is that, as the numerator is the dividend, if we multiply it we multiply the quotient; and, as the denominator is the divisor, if we divide it the quotient is multiplied.

1. Multiply $\frac{m-n}{3xy}$ by $m+n$.

MODEL SOLUTION.

OPERATION. $\frac{m-n}{3xy}\times(m+n)=\frac{m^2-n^2}{3xy}$.

EXPLANATION.—Since $m-n$ is divided by $3xy$, if I multiply it by $m+n$ and then divide (or indicate the division), I shall have $m+n$ times as large a quotient as at first. But the value of a fraction is the quotient of the numerator divided by the denominator. Hence multiplying the numerator of $\frac{m-n}{3xy}$ by $m+n$ I have $\frac{m^2-n^2}{3xy}$, which is $m+n$ times $\frac{m-n}{3xy}$.

2. Multiply $\frac{2mx}{3a^2b^2}$ by $3a$.

MODEL SOLUTION.

OPERATION. $\frac{2mx}{3a^2b^2}\times 3a=\frac{2mx}{ab^2}$.

EXPLANATION.—Since $2mx$ is to be divided by $3a^2b^2$, if I divide the divisor by $3a$, it will be contained in the dividend $3a$ times as many times as before. Hence $\frac{2mx}{ab^2}$ is $3a$ times $\frac{2mx}{3a^2b^2}$.

3. Multiply $\frac{3a}{2b}$ by $5a^2$.

4. Multiply $\frac{a-b}{3c+2}$ by $a+b$.

5. Multiply $\frac{1-x}{1-x^2}$ by $1-x$. *Prod.*, $\frac{1-x}{1+x}$.

6. Multiply $\frac{2c}{3a^2b}$ by $3a$.

7. Multiply $\frac{17x^2-4}{3ax+b}$ by ab.

8. Multiply $\frac{x^4-a}{3b+3c}$ by $3y$.

---

9. 15 is how many times $\frac{15}{3}$? a is how many times $\frac{a}{b}$? What effect does it have on a fraction to drop its denominator?

10. Multiply $\frac{3ax}{5}$ by $5a$.

Suggestion.—Multiply by 5 by dropping the denominator. Then multiply this product, $3ax$, by the other factor, a, of the multiplier.

11. As in the last, multiply $\frac{1-x}{1+x}$ by $(1-x^2)$. What are the factors of $1-x^2$?

69. Dropping the denominator *of a fraction multiplies the fraction by the number dropped.*

12. Multiply $\frac{m-n}{m+n}$ by $m^2+2mn+n^2$.

13. Multiply $\frac{m-n}{m+n}$ by $m^2-2mn+n^2$.

14. Multiply $\frac{abc}{5(x^4-y^4)}$ by x^2+y^2.

Case I. In Division.

70. RULE.—*To divide a fraction by an integer divide the numerator or multiply the denominator.*

DEM.—The reason for this may be given in two ways :

1st. Since division is the converse* of multiplication, we perform the converse operations, *i. e.*, to multiply a fraction by an integer we *multiply* the numerator or divide the denominator; hence to divide, we *divide* the *numerator* or multiply the denominator.

2d. As the numerator is the dividend, dividing it divides the value of the fraction, just as dividing the dividend divides the quotient; and as the denominator is the divisor, multiplying it divides the quotient.

1. Divide $\frac{ac}{c}$ by b.

MODEL SOLUTION.—Since the value of a fraction is the quotient of the numerator divided by the denominator, and dividing the dividend divides the quotient, I divide ac by c, and have $\frac{ac}{b}\div c=\frac{a}{b}$.

2. Divide $\frac{a}{b}$ by c.

---

* The pupil may need to be told that converse means about what he would mean by opposite, or reverse. The nice distinction need not be made, but he should use the right word.

Model Solution.—Since the value of a fraction is the quotient of the numerator divided by the denominator, and multiplying the divisor divides the quotient, I multiply b by c, and have $\frac{a}{b} \div c = \frac{a}{bc}$.

Note.—The pupil should be taught also to explain according to the first demonstration.

3. Divide $\frac{3a}{7x}$ by x.

4. Divide $\frac{3a}{7x}$ by a.

5. Divide $\frac{3a}{7x}$ by ax.

Suggestion.—Divide successively by the factors of ax, *i. e.*, divide by a by dividing the numerator, giving $\frac{3}{7x}$. Then divide this by x by multiplying the denominator, giving $\frac{3}{7x^2}$.

71. Always perform *multiplication and division in fractions* **as much by division as possible,** *this will keep the results in the lowest terms.*

6. Divide $\frac{6a^2}{13b^3}$ by $2a^2b$.

Multiply $\frac{6a^2}{13b^2}$ by $2a^2b$.

7. Divide $\frac{a^2-b^2}{x+y}$ by $a-b$.

Multiply $\frac{a^2-b^2}{x+y}$ by $a-b$.

5. Divide $\frac{a^2-b^2}{x+y}$ by $x-y$.

Multiply $\frac{a^2-b^2}{x+y}$ by $x-y$.

Case II. In Multiplication.

1. If the numerator of a fraction is $\frac{2}{3}$ and the denominator $\frac{5}{7}$, so that the fraction is $\frac{\frac{2}{3}}{\frac{5}{7}}$, and you multiply both numerator and denominator by 3×7, or 21, what does it become?

Does this process change the value of the fraction?

2. If a, b, c, and d, are integers, and you multiply both numerator and denominator of $\frac{\frac{a}{b}}{\frac{c}{d}}$ by bd, what does the fraction become?

Does this process change the value of the fraction?

3. How many times $\frac{3}{5}$ is 3? If, then, you want to multiply any number by $\frac{3}{5}$, if you first multiply by 3, do you get a product too great or too small?

How many times too great or too small?

How, then, will you get the true product from this?

4. If a and b are integers, and wishing to multiply any number by $\frac{a}{b}$, you first multiply by a, is your product too great, or too small? How much? Why?

6 times 5 is how many times as much as $\frac{6}{5}$ times 5?

72. RULE.—*To multiply by a fraction, multiply by the numerator and divide by the denominator.*

DEM.—Let it be required to multiply m, which is either an integer or a fraction, by $\frac{a}{b}$.

1st. Suppose a and b are both integers. Multiplying m by a gives a product b times as large as it should be, since we were to multiply by only a bth part of a; hence we divide the product, am, by b, and have $\frac{am}{b}$.

2d. When either a or b, or both, are fractions. Let c be the factor by which numerator and denominator of $\frac{a}{b}$ must be multiplied to make $\frac{a}{b}$ a simple fraction (see Exs. 1 and 2). Then will $\frac{ac}{bc}$ be a simple fraction, *i. e.*, ac and bc are each integral; and the multiplication is effected as in Case 1st, giving $\frac{acm}{bc}$. This reduced by dividing both terms by c gives $\frac{am}{b}$. Hence we see that in *any* case, to multiply by a fraction, we have only to multiply the multiplicand by the numerator of the multiplier, and divide this product by the denominator. It is also to be observed that this reasoning applies equally well whether the *multiplicand* is integral or fractional.

5. Multiply $\frac{a}{3b}$ by $\frac{5}{2b^3}$.

6. Multiply $\frac{a}{b}$ by $\frac{x}{y}$.

7. Multiply $\frac{c-d}{5a}$ by $\frac{c+d}{2-a}$.

8. Multiply $\frac{2x+3}{5}$ by $\frac{10x}{7}$.

9. Multiply $\frac{a^2-b^2}{3ab}$ by $\frac{2a^2}{a-b}$.

Suggestion.—Multiply $\frac{a^2-b^2}{3ab}$ by $2a^2$, by dividing the denominator by a, and multiplying the numerator by $2a$, thus obtaining their results $\frac{2a(a^2-b^2)}{3b}$, which is to be divided by $a-b$. To effect this, divide the numerator, thus obtaining $\frac{2a(a+b)}{3b}$, or $\frac{2a^2+2ab}{3b}$.*

10. Multiply $\frac{a^2-x^2}{x^2}$ by $\frac{a}{a+x}$. Also by $\frac{a}{a-x}$.

11. Multiply $\frac{a^2+b^2}{a^2-b^2}$ by $\frac{a-b}{a+b}$.

Case II. In Division.

1. How many times is $\frac{1}{3}$ contained in 1?
Are $\frac{2}{3}$ contained as many times in 1 as $\frac{1}{3}$ is? Why?
How many times are $\frac{2}{3}$ contained in 1?

2. How many times is $\frac{a}{b}$ contained in 1?

How many times is a contained in 1?

Then $\frac{a}{b}$ is contained how many times?

73. A fraction inverted, *i. e., its reciprocal, shows how many times it is contained in 1.*

---

* It is thought better to say nothing about *cancellation* till the *principle* of the rule is well established.

74. RULE.—*To divide by a fraction, invert the divisor and multiply the result by the dividend.*

DEM.—Let $\frac{a}{b}$ be any fraction, and suppose we wish to find how many times it is contained in c (c may be an integer, fraction, or mixed number). Now $\frac{a}{b}$ is contained in 1 $\frac{b}{a}$ times; hence it is contained in c, c times $\frac{b}{a}$ times, or $\frac{bc}{a}$ times.

3. Divide $\frac{3a}{5x}$ by $\frac{m}{n}$.

4. Divide $7y^2$ by $\frac{10ax}{3by}$. *Quot.* $\frac{21by^3}{10ax}$.

5. Divide 6 by $\frac{2a}{3b}$.

6. Divide $\frac{a}{5}$ by $\frac{3}{x}$.

7. Divide $\frac{2-c}{3}$ by $\frac{4}{2+c}$.

8. Divide $\frac{a-1}{c+d}$ by $\frac{m-n}{x+y}$.

9. Divide $\frac{x^2-2xy+y^2}{ah}$ by $\frac{x-y}{bc}$.

10. Divide $\frac{2x^2}{a+b}$ by $\frac{3x^3}{a^2-b^2}$.

Complex Fractions.

1. If you multiply numerator and denominator of $\dfrac{\frac{2a}{3b}}{\frac{x}{2y}}$ by $6by$, what will be the result?

How would you multiply $\frac{2a}{3b}$ by $6by$?

How multiply by the factor $3b$ (69)?

How by the factor $2y$?

How would you multiply $\frac{x}{2y}$ by $6by$?

How by the factor $2y$?

How by the factor $3b$?

75. RULE.—*To reduce a mixed fraction to a simple one, multiply both numerator and denominator by the lowest common multiple of the denominators of the partial fractions.*

DEM.—This process removes the partial denominators, since each fraction is multiplied by its own denominator, at least, and this is done by dropping the denominator. It does not alter the value of the fraction, since it is multiplying dividend and divisor by the same quantity.

2. Reduce $\dfrac{\frac{3cy}{5a^2x}}{\frac{2b}{7axy}}$ to a simple fraction, and put the result in its lowest terms.

3. Reduce $\dfrac{\frac{ax+b}{a}}{\frac{bx-a}{b}}$ to a simple fraction.

4. Reduce $\dfrac{\frac{2a^2}{3b}}{\frac{2a}{5b^3}}$ to a simple fraction.

5. Reduce $\dfrac{\frac{a^2-b^2}{x^2-y^2}}{\frac{a+b}{x-y}}$ to a simple fraction.

Suggestion.—In some cases it is better to conceive the fraction as a problem in division, and work accordingly. Thus, in this case, we have $\dfrac{a^2-b^2}{x^2-y^2} \div \dfrac{a+b}{x-y} = \dfrac{a-b}{x+y}$.

76. It is *also well to remember that as division is the converse of multiplication, we may divide one fraction by another by dividing the numerator of the dividend by the numerator of the divisor, and the denominator of the dividend by the denominator of the divisor.*

6. Reduce $\dfrac{\frac{x^2-2xy+y^2}{x^3-y^3}}{\frac{x^2+xy+y^2}{x-y}}$ to a simple fraction.

Suggestion.—Use the preceding suggestion and (76).

Cancellation.

77. Multiplication and division of fractions *is greatly facilitated by canceling any factor which would, in the final result, appear both in the numerator and denominator.*

This is the same principle as the pupil has become familiar with in arithmetic. The reason for the process is that dropping a factor from a number is the same as dividing the number by that factor; hence, dropping the same factor from numerator and denominator is the same as dividing both terms by that factor (58).

Perform the following by canceling as much as possible:

1. Multiply $\frac{15a^2x^3}{24c^2y}$ by $\frac{6cy^2}{5bx^3}$.

2. Divide $\frac{3ax^2-3ay^2}{15cz^3}$ by $\frac{a^2x+a^2y}{5cz}$.

3. Multiply $\frac{a^2-x^2}{2}$ by $\frac{2a}{a-x}$.

4. Multiply $\frac{a+x}{a}$, $\frac{a-x}{x}$, and $\frac{a^2-x^2}{a^2+x^2}$, together.

5. Divide $\frac{6(a+x)+2}{3}$ by $\frac{4}{3(a-x)}$.

6. Multiply $\frac{x^2-y^2}{x}$, $\frac{x}{x+y}$ and $\frac{1}{x-y}$ together.

7. Divide $\frac{a^4-2a^2x^2+x^4}{a^3x+ax^3}$ by $\frac{a^2-x^2}{a^2+x^2}$.

Mixed Numbers.

78. Mixed numbers *may be reduced to improper forms, and then operated upon according to the common rules for addition, subtraction, multiplication, and division. But in adding and subtract-*

ing it is usually better not to do so, but to combine the integral and fractional parts separately, as given in the suggestions under Exs. 14 and 12, pages 72, 75.

1. Multiply $1+\frac{1}{x^2}$ by $x-\frac{1}{x}$. *Prod.*, $\frac{x^4-1}{x^3}$.

2. Divide $ab+\frac{b}{a}$ by $\frac{1}{a^2-1}$.

3. Divide 12 by $\frac{(a+x)^2}{x}-a$.

4. Multiply $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$ and $a+\frac{ax}{a-x}$ together.

Simple Equations

SECTION XIV.

How Problems are Solved in Algebra.

1. John is 3 times as old as James, and the sum of their ages is 32; how old is each?

SOLUTION.—This example is a very simple one, and can easily be solved mentally. Thus, we see that since John's age is 3 times James's, both their ages together make 4 times James's age, and this is 32 years. Now 4 times James's age = 32 years. Hence, James's age is ¼ of 32, or 8 years; and John's age being 3 times James's, is 3×8, or 24 years.

To solve this by Algebra, we proceed as follows: Let x represent James's age; then, since John is 3 times as old, $3x$ will represent his age; and the sum of their ages will be $3x+x$. Now the statement is that the sum of their ages is 32, hence $3x+x=32$.

Or, what is the same thing, $4x=32$. If $4x=32$, $x=\frac{1}{4}$ of 32, or $x=8$. But x stood for James's age; hence, James's age is 8, and John's being 3 times as much is 3×8, or 24.

79. **The expression $3x+x=32$** *is what is called an* **Equation**; *and it is by the use of equations that we solve problems in Algebra.*

2. A merchant said that he had 72 yards of a certain kind of cloth, in three rolls. In the first roll, there were a certain number of yards; in the second, 3 times as many as in the first; and in the third, twice as many as in the first. How many yards were there in the first roll?

Suggestion.—The equation is $x+3x+2x=72$.
Now, $x+3x+2x$ is $6x$, hence $6x=72$.
And if $6x=72$, x, or $1x$, is $\frac{1}{6}$ of 72. $x=12$.

Queries.—What does the x stand for? Answer. The number of yards in the first roll. In this problem, which is the most, $x+3x+2x$, or 72? To start with do you know how much x is? Then is it a *known*, or an *unknown*, quantity at the outset?

---

80. **The number** *which we desire to find as the answer of a problem is represented in the beginning of the solution by one of the latter letters of the alphabet, usually* **x**, *if there is need of but one letter, and is called the* **Unknown Quantity.**

3. A boy on being asked how old he was, replied, "if you add to my age 3 times my age, and 5 times my age, and subtract twice my age, the result will be 49 years." How old was he?

Suggestion.—The equation is $x+3x+5x-2x=49$.
Hence, since $x+3x+5x-2x$ is $7x$, $7x=49$.
If $7x=49$, $x=\frac{1}{7}$ of 49, or $x=7$.

4. There are three times as many girls as boys in a party of 60 children. How many boys are there? How many girls?

5. In a barrel of sugar weighing 200 lbs., there are three varieties, A, B, and C, mixed. There is 7 times as much of B as of A, and twice as much of C as of A. How much of A is there? How much of each of the other kinds?

Ans., of A, 20 lbs.; of B, 140 lbs.; of C, 40 lbs.

81. The part *of an equation on the left of the sign* = *is called the* **First Member,** *and that on the right, the* **Second Member.**

6. There were 4 kinds of liquors put into a cask, 2 times as much of the second as of the first, 2 times as much of the third as of the second, and 2 times as much of the fourth as of the third. The cask sprung a leak, and three times as much leaked out as was put in of the first kind, when it was found that there were 36 gallons remaining. How much was there put in of each kind?

Suggestion.—The equation is $x+2x+4x+8x-3x=36$.

---

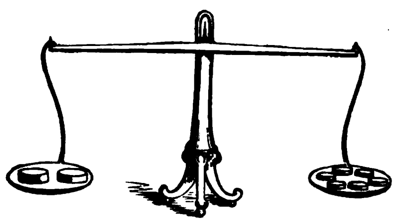
7. In a pasture there are a certain number of cows and 23 sheep, in all 34 animals. How many cows were there?

Solution.—As it is the number of cows we seek, let x represent the number of cows. Then $x+23$ is the number of animals in the pasture, and the equation is

$$x + 23 = 34.$$

Now the $x+23$ means just the same thing as the 34, that is, $x+23=34$. So, if we subtract 23 from each, there will be just as much left of one as of the other. Subtracting 23 from $x+23$, there remains x, and subtracting 23 from 34, there remains 11. Hence $x=11$. Now as x represented the number of cows, we know that there were 11 cows.

82. Illustration.—An Equation may be compared to a pair of scales with equal arms.



First Member = Second Member.

The weights in the scale pans represent the members of the equation ; and, as there must be just as much in one pan as in the other, in order that the scales should balance, so the amount in one member of an equation must be just the same as the amount in the other, in order that the equation should be true.

Again, when the scales are balanced by the weights in the two pans, if we take 3 ounces out of one pan, we must *take just as much* out of the other, or the scales will not balance ; or, if we put any amount more into one pan, we must add just as much to the weights in the other pan, or the scales will not balance.

If the scales are balanced with weights, and we change the weights on one side so as to have 5 times as much, how must we change the other side to preserve the balance?

If we divide what is in one pan, and leave only $\frac{1}{6}$ as much in it, what must we do to the contents of the other pan to preserve the balance?

Now, it is just the same with an equation. If we make a true equation, there is just as much on one side as on the other; and if we add anything to one member, we must add *just as much* to the other; or, if we subtract anything from one member, we must subtract *just as much* from the other. If we multiply one member by 5, what must we do to the other to keep the equation true? If we divide one member of an equation by 7, what else must we do to keep the equation true?

8. In a certain pasture there are three times as many horses as cows, and 20 sheep. In all there are 100 animals. How many cows are there? How many horses?

Suggestion.—The equation is $x+3x+20=100.$
Subtracting 20 from each member, $x+3x=80.$
Uniting the terms of the first member, $4x=80.$
Dividing each member by 4, $x=20.$
Hence there were 20 cows; and, as there were three times as many horses as cows, there were 60 horses.

NOTE.—Observe that, after we have made the equation, we want to put it into such a form that x shall *stand alone* in the first member, and then it will tell what the value of x is. Thus, $x=20$ says that x is 20, which is just what we wanted to know.

Again, when we had $x+3x+20=100$, in order to have x alone in the first member, we subtracted the 20 from this member. Can you tell why we subtracted it from the second member also? If we had subtracted the 20 from the first member only, would the equation have been true afterwards? Which member would have been the larger? How much? How then could we make it true?

Once more, when we had $4x=80$, can you tell why we divided the first member by 4? Why the second?

9. In a basket of 60 apples there are 4 times as many red apples as yellow, and 10 green apples. How many yellow apples are there? How many red?

10. John and James together have 75 cents. James has 25 cents less than John. How many cents has John?

Suggestion.—Let x represent the number of cents which John has. Then, as James has 25 cents less, $x-25$ will represent what he has. But both together have 75 cents. Hence the equation is

$$x+x-25=75.$$

Now, if we drop the -25 from the first member, we make this member 25 greater than it now is, *i. e.*, $x+x$ is 25 greater than $x+x-25$. Therefore, if we add 25 to the second member, making it 100, the members will still be equal.

This gives $x+x=100$,

or, $2x=100$.

Hence $x=50$, the number of cents which John has.

11. A merchant has 90 yards of cloth in two pieces. The longer piece lacks ten yards of containing 3 times as much as the shorter. How much in each piece?

Suggestion.—The equation is $x+3x-10=90$. What do we want to stand alone in the first member? If we drop -10 from the first member, does that member become more or less? What then must we do to the second member? Why?

12. Divide the number 50 into two parts so that one part shall lack 10 of being 5 times the other.

Suggestion.—The parts are represented by x, and $5x-10$. They are 10 and 40.

13. Divide the number 50 into 3 parts, such that the second shall be 5 more, and the third 15 less than the first.

Suggestion.—The equation is $x+x+5+x-15=50$. The parts are 20, 25, and 5.

14. There are 52 animals in a field. Twice the number of cows + 11 is the number of sheep, and 3 times the number of cows − 13 is the number of horses. How many of each kind?

Ans., 9 cows, 29 sheep, and 14 horses.

---

15. A man said of his age,

"If to my age there added be
Its half, its third, and three times three,
Six score and ten the sum will be;
What is my age? Pray show it me."

Suggestion.—The equation is

$$x+\frac{x}{2}+\frac{x}{3}+9=130.$$

Subtracting 9 from each member

$$x+\frac{x}{2}+\frac{x}{3}=121.$$

Now, we can get rid of the fractions in the first member by multiplying it by 6, the product of both the denominators. Thus, 6 times the first member is $6x+3x+2x$. Then, if we also multiply the second member by 6, the products will be equal. For if two quantities are equal,* 6 times one of them is equal to 6 times the other. Hence we have $6x+3x+2x=726$.

Uniting terms, $11x=726$.

Dividing by 11, $x=66$.

16. Mary gave half her books to Jane, and one-third of them to Helen, when she had but two left. How many had she at first?

Suggestion.—Let x represent the number of books Mary had

* In this case the two quantities are the two members.

at first. Then she gave Jane $\frac{x}{2}$, and Helen $\frac{x}{3}$ books. And what she gave the other girls, added to what she had left, makes all she had in the first place. Hence the equation is

$$\frac{x}{2}+\frac{x}{3}+2=x.$$

Multiplying each member by 6, $3x+2x+12=6x$.
Subtracting $5x$ from each member, $12=x$, or $x=12$.
That is, Mary had 12 books at first. Pupil give the the reasons. Thus, why multiply the first member by 6? Why the second? Why subtract $5x$ from the first member? Why from the second?

17. A boy lost 25 cents of some money which his uncle gave him, and gave half he had left to his brother. He then earned 50 cents, when he had just as much as his uncle gave him. How much did his uncle give him?

Suggestion.—Let $x=$* the number of cents his uncle gave him. Then he had $x-25$ cents after losing 25 cents. After giving away half of this, he had the other half, or $\frac{x-25}{2}$ cents, left. He then earned 50 cents, and the amount he had was equal to what his uncle gave him.

Hence the equation is $\frac{x-25}{2}+50=x$.
Multiplying each member by 2, (Why?) $x-25+100=2x$.
Uniting, $-25+100$ makes 75, and $x+75=2x$.
Subtracting x from each member, (Why?) $75=x$, or $x=75$.

NOTE.—The pupil must not think that because these examples are so simple that he can "do them in his head" without any algebra, and may be with less work, that therefore algebra is a very clumsy method of solving examples. He will find by and by, that though the equation does not really help any in the solution of such simple questions, it will solve a great many very difficult ones about which he might puzzle his brains a great while to no purpose, if

* The sign of equality used in this way means the same as the word "represent."

algebra did not come to his aid. Stick to it, then, and learn how to use this new instrument, the *Equation*, and you will by and by find it wonderfully useful. It is a grand patent for solving problems.

N. B.—*Be careful to give the reason for everything you do to each member of an equation.*

18. A boy, being asked how many marbles he had, said, "If I had five more than I have, half the number subtracted from 30 would leave twice as many as I now have." How many marbles had he?

Suggestions.—Letting x represent the number of marbles the boy had, the equation is

$$30-\frac{x+5}{2}=2x.$$

Now there is a little peculiarity about this equation, which the pupil must be careful to notice whenever it occurs, or he will make a great many mistakes. It is this: When we multiply both members by 2, to get rid of the fraction, we must write $60-x-5=4x$. The mistake would be to write $60-x+5=4x$. The explanation is, that the $-$ sign before $\frac{x+5}{2}$ shows that it is to be subtracted from 30, hence the signs of the terms composing it, viz., x and 5, must be changed, according to the rule for subtraction. But the pupil may think that we do not change the sign of the x. If he does he mistakes. The $-$ sign before the fraction $\frac{x+5}{2}$ does not belong to the x, but to the fraction *as a whole.* The sign of x in the fraction $-\frac{x+5}{2}$ is $+$, since when no sign is expressed $+$ is understood. What then becomes of the $-$ sign before the fraction, if it is not the same as the sign of x in the equation $60-x-5=4x$? It has been dropped, since the thing signified by it has been performed, and the $-$ sign before the x, in $60-x-5$, is the

sign of that term in the original equation changed. The boy had 11 marbles.

19. What is the value of x in the equation $3x-\frac{2-2x}{3}=21\frac{1}{3}$?

***Suggestion.*—Multiplying each member by 3, we have $9x-2+2x=64$. Hence $x=6$.**

---

20. Two boys were to divide 32 marbles between them so that $\frac{1}{2}$ of what one had should be 5 less than what the other had. How many was each to have?

***Suggestion.*—Letting x = what one had,**

then $32-x$ = what the other had.

The equation is $\frac{x}{2}+5=32-x$,

or $\frac{32-x}{2}+5=x$.

***Query.*—Why will either equation answer the purpose?**

21. What number is that to which if 7 be added, half the sum will exceed $\frac{1}{5}$ of the remainder of the number after 3 has been subtracted, by 8?

Equation, $\frac{x+7}{2}-\frac{x-3}{5}=8$. Hence $x=13$.

22. The sum of two numbers is sixteen, and the less number divided by three is equal to the greater divided by five. What are the numbers?

***Suggestion.*—Let x and $16-x$ represent the numbers.**

23. Divide twenty-two dollars between A and B, so that if one dollar be taken from three-fourths of B's share, and

three dollars be added to one-half of A's money, the sums shall be equal. How many dollars will each have?

Ans., A will have $10, and B $12.

24. The sum of two numbers is thirty-three. If one-sixth of the greater be subtracted from two-thirds of the less number, the remainder will be seven. What are the numbers?

25. The sum of A's and B's money is thirty-six dollars. If five-eighths of B's, less two dollars, be taken from three-fourths of A's, the difference will be seven dollars. How many dollars has each?

26. The difference between two numbers is twenty-five; and if twice the less be taken from three times the greater, the remainder will be eighty? What are the numbers?

27. A and B gain money in trade, but A receives ten dollars less than B. If A's share be subtracted from twice B's, the remainder will be fifty-seven dollars. How much money did each receive?

28. One number is four less than another, and if twice the less be subtracted from five times the greater, the remainder will be thirty-eight. What are the numbers?

29. Two farms belong to A and B. A has twenty acres less than B. If twice the number of acres in A's farm be taken from three times the number in B's, the remainder will be one hundred. How many acres has each?

30. One number is seven less than another, and if three times the less be taken from four times the greater, the

remainder will be six times the difference between the two numbers. What are the numbers?

31. Anna is four years younger than Mary. If twice Anna's age be taken from five times Mary's, the remainder will be thirty-five years. What is the age of each?

32. One number is ten less than another. If three times the less be taken from five times the greater, the remainder will be seven times the difference of the two numbers. What are the numbers?

Simple Equations

THEIR TRANSFORMATION.

SECTION XV.

83. We have seen in the preceding Section what an *Equation* is, and how it is used in solving problems. We have also seen that the solution of a problem by an equation consists of two parts, 1st, making the equation, and 2d, so changing the equation that the unknown quantity shall stand alone in one member, the other member being all known quantities. The first part is called the *Statement* of the problem, and the second part is called the *Solution of the Equation.* We propose in this section to give attention to the *Solution of Equations.* It will be but a formal statement of what has been taught in the preceding section, with examples for practice. You have seen that after getting the statement in an equation we have to change the form of the equation very much before we get the answer, or value of the unknown quantity. Thus in Ex. 17, page 94, we had $\frac{x-25}{2}+50=x$, at the first. Then we changed it to $x-25+100=2x$; then this to $x+75=2x$; then this to $75=x$, or $x=75$. These changes are called *Transformations.*

84. **To Transform** *an Equation is to change its form without destroying the equality of the members.*

85. *There are* **four principal** *transformations of simple equations containing one unknown quantity, viz.:* **Clearing of Fractions, Transposition, Collecting Terms,** *and* **Dividing by the co-efficient of the unknown quantity.**

86. These transformations are all based on two simple principles called *axioms.*

87. **An Axiom** *is a proposition which states a principle that is so simple, elementary and evident, as to require no proof.*

Axioms.

AXIOM 1. *Any operation may be performed upon any term or upon either member, which does not affect the value of that term or member, without destroying the Equation.*

AXIOM 2. *If both members of an Equation are increased or diminished alike, the equality is not destroyed.*

We shall now proceed to consider formally these several transformations.

FIRST TRANSFORMATION.

To Clear an Equation of Fractions.

88. RULE.—*Multiply both members by the least or lowest common multiple of all the denominators.*

DEM.—This process clears the Equation of fractions, since, in the process of multiplying any particular fractional term, its denominator is one of the factors of the L. C. M. by which we are multiplying ; hence dropping the denominator multiplies by this factor, and then this product (the numerator) is multiplied by the other factor of the L. C. M.

This process does not destroy the Equation, since both members are increased or diminished alike.

1. Clear the equation $3+\frac{x}{3}-\frac{x}{6}=\frac{3x-26}{2}$ of fractions.

MODEL SOLUTION.*

OPERATION.—The equation is $3+\frac{x}{3}-\frac{x}{6}=\frac{3x-26}{2}$. Multiplying both members by 6 I have $18+2x-x=9x-78$. I multiply by 6 because 6 being the L. C. M. of all the denominators, I can drop the denominator of each fraction when I multiply. Thus to multiply $\frac{x}{3}$ by 6, I multiply by the factors of 6, 3 and 2. To multiply $\frac{x}{3}$ by 3 I drop its denominator (69), and then multiply this result, x, by 2, and have $2x$. To multiply $-\frac{x}{6}$ by 6 I have only to drop its denominator. Thus 6 times the first member is $18+2x-x$.

* These *Model Solutions* are designed as specimens of the explanations which should be given by the pupil after he has solved the example on the blackboard. Be sure of the *Why*, as well as *How*.

Now, as I have multiplied the first member by 6, I must multiply the second by 6, according to AXIOM 2, in order to keep the members equal. To multiply $\frac{3x-26}{2}$ by 6, I multiply by 2 and 3, the factors of 6. I multiply by 2 by dropping the denominator 2 (69) and then multiply this result, $3x-26$, by 3, obtaining $9x-78$. Hence the equation *cleared of fractions is* $18+2x-x=9x-78$.

Ex. 2—5. Clear the following of fractions, being careful to explain the process and give the **reason** for each step as in the *Model Solution* **above:**

2. $\frac{x}{4}+24=\frac{3x}{2}$.

3. $\frac{x}{2}+\frac{x}{3}=15-\frac{x}{4}$.

4. $\frac{x-5}{4}+6x=\frac{284-x}{5}$.

5. $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$. See Ex. 18, page 95:

89. Whenever *in clearing an equation of fractions, the denominator of a fraction having a polynomial numerator, and preceded by the − sign, is dropped, the signs of all the terms of the numerator must be changed.**

Ex. 6—9. Clear the following of fractions, being careful to notice the application of (89), giving the reason for the change of signs as in Ex. 18, page 95:

* This is really the case when the numerator is a monomial as well. Thus in Ex. 3, when we multiply $13-\frac{x}{4}$ by 12, we may properly consider the x, as $+x$, and when we write $156-3x$, consider the $-3x$ as three times the numerator *with its sign changed.*

6. $\frac{12x+26}{5}-2x=15-\frac{x+3}{3}$.

7. $\frac{3x+4}{5}-\frac{7x-3}{2}=\frac{x-16}{4}$.

8. $x-\frac{3x-5}{2}=12-\frac{2x-4}{3}$.

9. $\frac{19-x}{2}=x+\frac{11-x}{3}$.

Query.—Should you change the signs of the numerator of $\frac{11-x}{3}$ when you drop its denominator? Why? If the second member were $x-\frac{11-x}{3}$ would you change the signs of *both* 11 and x? What is the sign of 11 in $x+\frac{11-x}{3}$, and in $x-\frac{11-x}{3}$? *Answer.* It is + in both cases.

10. Clear of fractions $\frac{x}{a}+\frac{2a+x}{3b}=ax-\frac{x}{2b}$.

Suggestion.—The multiplier is $6ab$. The equation cleared of fractions is $6bx+4a^2+2ax=6a^2bx-3ax$.

11. Clear of fractions $x+\frac{ax}{c}=b-\frac{dx}{c}$.

12. Clear of fractions $bc-\frac{ab}{x}=-d-\frac{1}{x}$.

13. Clear of fractions $\frac{x}{a-b}+\frac{a-x}{3}=\frac{ab}{a+b}$

Suggestion.—The L. C. M. is $3a^2-3b^2$. The equation, when cleared, is $3ax+3bx+a^3-a^2x-ab^2+b^2x=3a^2b-3ab^2$.

SECOND TRANSFORMATION.

To Transpose a Term.

90. Transposing *a term is changing it from one member of the equation to the other without destroying the equality of the members.*

91. RULE.—*To transpose a term, drop it from the member in which it stands and insert it in the other member with the sign changed.*

DEM.—If the term to be transposed is +, dropping it from one member diminishes that member by the amount of the term, and writing it with the − sign in the other member, takes its amount from that member; hence both members are diminished alike, and the equality is not destroyed. (Repeat AXIOM 2.)

2nd.—If the term to be transposed is −, dropping it *increases* the member from which it is dropped, and writing it in the other member with the + sign *increases* that member by the same amount; and hence the equality is preserved. (Repeat AXIOM 2.)

Ex. 1. Given $5x+4+3x-3=13-2x$ to transpose, so that all the terms containing the unknown quantity, x, shall stand in the first member, and the known terms in the second.

MODEL SOLUTION.

OPERATION. $5x+4+3x-3=13-2x$,

$$5x+3x+2x=13-4+3.$$

EXPLANATION.—Dropping 4 from the first member diminishes that member by 4; hence to preserve the equality, I subtract 4 from the second member, *or indicate* the subtraction by writing it in the second member with the − sign. Thus the term 4 is transposed.

Dropping −3 from the first member *increases* that member by

3; and hence to preserve the equality I add 3 to the second member. Thus the term -3 is transposed.

Dropping $-2x$ from the second member *increases* that member by $2x$, hence I increase the first member by adding $2x$ to it, and thus preserve the equality.

I have thus arranged the terms so that all those containing the unknown quantity stand in the first member, and all known terms in the second member; and yet I have preserved the equality. This is called transposition.

Ex. 2—14. The following exercises are the equations under the preceding transformation, cleared of fractions. Carry forward the work of reducing them by transposition, *i. e.*, put them in such forms that only unknown terms shall be in the first member, and only known terms in the second.

2. $18+2x-x=9x-78$.

3. $x+96=6x$.

4. $6x+4x=156-3x$.

5. $5x-25+120x=1136-4x$.

6. $6x+6+4x+8=192-3x-9$.

7. $36x+78-30x=225-5x-15$.

8. $12x+16-70x+30=5x-80$.

9. $6x-9x+15=72-4x+8$.

10. $57-3x=6x+22-2x$.

11. $6bx+4a^2+2ax=6a^2bx-3ax$.

Suggestion.—When the proper transpositions have been effected, all the terms will be in the first member except $4a^2$. Why?

12. $cx+ax=bc-dx$.

13. $bcx-ab=-dx-1$.

14. $3ax+3bx+a^3-a^2x-ab^2+b^2x=3a^2b-3ab^2$.

Query.—Which terms will be in the first member? Which in the second? Why?

THIRD TRANSFORMATION.

Collecting Terms.

92. This transformation *is so simple as to need little explanation. Thus, when we have*

$$6x+4x-9x=72+8-15$$

that we may add $6x+4x-9x$ and call the first member x is evident from the first axiom, since this transformation does not change the value of this member. In like manner $72+8-15$ is the same as 65, so that if the equation $6x+4x-9x=72+8-15$ is true (*i. e.*, if the members are equal), $x=65$ is true, for the members of the latter have the same value as those of the former.

Ex. 1–4. Collect the terms in the following, and give the reason why the equation is not destroyed: *

* We often hear about "changing the *value* of an equation," "dividing an equation," etc., expressions which are grossly erroneous. *An equation* is not a quantity—is not a thing which can be said to have value, be multiplied or divided. The *members* are quantities, and these can be multiplied, divided, etc. Instead of "changing the value of the equation," the thing meant is "de

1. $6x-2x+3x=57-22$.

2. $6x+4x+3x=192-9-6-8$.

3. $70x-12x+5x=16+30+80$.

4. $cx+dx=bc-ab$.

Suggestion. $cx+dx$ can be added with reference to x (see 87), and becomes $(c+d)x$; and, in like manner $bc-ab$ can be added with reference to b, becoming $b(c-a)$. Hence we have $(c+d)x=b(c-a)$.

5. Collect the terms of $2a^2x-x=a^2+a^3$.

Result, $(2a^2-1)x=a^2(1+a)$.

6. Collect the terms of $x+cx=a^2c-a^2b+a^2$.

7. Collect the terms of $a^2x+2abx+b^2x=a^2-2ab+b^2$. The result should be written $(a+b)^2x=(a-b)^2$.

FOURTH TRANSFORMATION.

Dividing by the Co-efficient of the Unknown Quantity.

93. *This also is a simple operation, and needs little attention.* If we have $4x=52$, we can divide $4x$ by 4, obtaining x, and preserve the equality of the members by dividing 52 by 4 also. Hence, if $4x=52$, $x=13$, since

---

stroying the equality of the members." Thus, when we multiply the members of an equation, it is absurd to say we have multiplied the equation, for the reason given above, and it is equally absurd to say that it does not change the *value* of the equation. We *have* changed the value of the terms and members—the only things which have value, but we have so changed them as not to destroy the equation, *i.e.*, the equality between the members.

x is $\frac{1}{4}$ of $4x$, and 13 is $\frac{1}{4}$ of 52. That this transformation does not destroy the equation is evident from Axiom 2.

Ex. 1—4. Divide both members of the following by the coefficients of the unknown quantity:

1. $7x=35$.
2. $13x=169$.
3. $63x=126$.
4. $(c+b)x=b(c-a)$.

Result in last, $x=\dfrac{b(c-a)}{c+b}$.

Ex. 5—7. Divide both members of the following by the co-efficient of the unknown quantity:

5. $(2a^2-1)x=a^2(1+a)$.
6. $(1+c)x=a^2(c-b+1)$.
7. $(a+b)^2x=(a-b)^2$.

---

94. It frequently *happens that we wish to change the signs of the terms of one member of an equation. This can be done if we change the signs of the terms of the other member also.*

1. Change the signs of all the terms of $-6x+9x-4x=-72-18+15$, and show that it does not destroy the equation.

Suggestion.—The equation, with its signs changed, is $6x-9x+4x=72+18-15$. This can be obtained from $-6x+9x-4x=-72-18+15$ by multiplying, or dividing, both members of the latter by -1. Hence if $-6x+9x-4x=-72-18+15$, $6x-9x+4x=72+18-15$, according to AXIOM 2.

Ex. 2—7. Change the signs in the following, and give the reason why the equation is not destroyed thereby:

2. $-4x=-20.$	5. $-x=-5.$
3. $-2x-3x=-70+10.$	6. $-x=10.$
4. $7x-12x=-100.$	7. $-(a-b)x=-(a^2-b^2).$

Suggestion.—In $-(a-b)x=-(a^2-b^2)$ we must remember that $(a-b)$, and (a^2-b^2), being in parentheses, are to be treated as single quantities (84). Hence the equation, with its signs changed, is $(a-b)x=a^2-b^2$.

8. Change the signs of $(-1+a)x=a(c-b)$.

Result, $(1-a)x=a(b-c)$.

9. Change the signs of $-(-a+b)x=\frac{-2a}{a+b}$.

Result, $(b-a)x=\frac{2a}{a+b}$.

Simple Equations

WITH ONE UNKNOWN QUANTITY.

SECTION XVI.

95. To Solve *an Equation is to find the value of the unknown quantity: that is, to find what value it must have in order that the equation be true.*

ILL.—In the equation $4x-2=2x+4$, if we call x, 3, the first member is 10, and as the second is also 10 for this value of x, the equation is true. But if we try any (or every) other number than 3 for x, we shall find that the equation will not be true. Thus trying 4 for x, we find the first member 14 and the second 12; and the equation is not true. Again, try 5. The first member becomes 18 and the second 14, and the equation is not true.

Let the student see if he can ascertain by *inspection* what is the value of x in the following:

$$x+3=3x+1.$$

Though this equation is very simple, it is probable that it will take the student some time to *guess* out the value of x which makes it true. Thus, 2 makes the first member 5 and the second 7; and the equation is not true for this value. If he tries 3 the result is worse than before. But 1 makes each member 4, and for this value of x the equation is true, *and for no other.*

But, if it is so difficult to hit upon just the value of x which is required to make so simple an equation true, the task would be quite hopeless in such an one as

$$\frac{3x-1}{5}-\frac{13-x}{2}=\frac{7x}{3}-\frac{11(x+3)}{6}.$$

Yet we have seen that there is a very simple method of solving any such equation, so as to tell certainly and easily what the value of x is. This process is called *Solving the Equation*, or sometimes, the *Resolution of the Equation.* Familiarity with it is very important, and facility can be acquired only by practice.

96. One thing *must be fixed in the pupil's mind at the outset, viz., that he can make but two classes of changes upon an equation: Such as do not affect the value of the members, or such as affect both members equally. Every operation must be seen to conform to these conditions.*

To Solve a Simple Equation with one Unknown Quantity.

97. RULE.—1. *If the equation contains fractions, clear it of them by Art. 88.*

2. *Transpose all the terms involving the unknown quantity to the first member, and the known terms to the second member by Art. 90.*

3. *Unite all the terms containing the unknown quantity into one by addition, and put the second member into its simplest form.*

4. *Divide both members by the co-efficient of the unknown quantity.*

DEM.—The first step, clearing of fractions, does not destroy the equation, since both members are multiplied by the same quantity (AXIOM 2).

The second step does not destroy the equation, since it is adding the same quantity to both members, or subtracting the same quantity from both members (AXIOM 2).

The third step does not destroy the equation, since it does not change the value of the members (AXIOM 1).

The fourth step does not destroy the equation, since it is dividing both members by the same quantity, and thus changes the members alike (AXIOM 2).

Hence, after these several processes, we still have a true equation. But now the first member is simply the unknown quantity, and the second member is all known. Thus we have *what the unknown quantity is equal to; i. e., its value.*

1. Solve the equation $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{8}=\frac{6x+7}{8}$.

MODEL SOLUTION.

OPERATION.*—(1) . . . . $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{8}=\frac{6x+7}{8}$,

(2) . . . . $20x+20+24x-32+5=30x+35$,

(3) . . . . $20x+24x-30x=35-20+32-5$,

(4) . . . . $14x=42$,

(5) . . . . $x=3$.

* This is a sample of what is to be written on the black-board.

EXPLANATION.*—I first clear equation (1) of fractions by multiplying both members by the L. C. M. of its denominators, which is 40. This does not destroy the equation (AXIOM 2). I multiply $\frac{x+1}{2}$ by 40 by dropping its denominator 2, thus getting $x+1$, as the result of multiplying by 2, and then multiply $x+1$ by 20, getting $20x+20$. [In like manner explain the entire process of clearing of fractions.]

Having cleared the equation of fractions I have (2). I now transpose the terms containing x to the first member and the known terms to the second member. Thus, dropping $30x$ from the second member diminishes it by that amount, whence to preserve the equality of the members I subtract $30x$ from the first member, *i. e.*, write it in that member with its sign changed. [In like manner explain the transposition of each term.]

I know equation (3) to be true, since I have changed both members alike, that is, have added to and subtracted from both members the same quantities. I now add together the terms of the first member, *which does not affect the value of the member*, and have $14x$. In the same manner uniting the terms of the second member does not alter its value; hence $14x=42$. Finally, I divide both members by 14 and have $x=3$. This operation does not destroy the equation, since *both* members of the equality $14x=42$ are divided by the same number (AXIOM 2). Hence 3 is the value of x.

Ex. 2—5. Solve the following equations:

2. $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}$. $x=12$.

3. $\frac{x-5}{4}+6x=\frac{284-x}{5}$. $x=9$.

4. $\frac{x}{4}+23=\frac{3x}{2}$. $x=18\frac{2}{5}$.

* This is a sample of what is to be given by the pupil when he explains the work on the black-board.

5. $\frac{6x-10}{3}=\frac{18-4x}{3}+x.$ $x=4.$

Suggestion.—Let us put 4 in the place of x in the last equation, and see if it makes the members equal. Thus $\frac{6\times4-10}{3}$ is $\frac{24-10}{3}$, or $\frac{14}{3}$. Again, $\frac{18-4\times4}{3}+4$ is $\frac{18-16}{3}+4$, or $4\frac{2}{3}$. Hence, 4, when put for x, makes the equation $\frac{14}{3}=4\frac{2}{3}$, which is evidently true.

98. Putting one *quantity in the place of another, as the* **4** *for the* **x** *in this suggestion, is called* **Substituting.**

99. An equation *is said to be* **Satisfied** *for a value of the unknown quantity which makes it a true equation: i. e., which makes its members equal.*

100. To Verify *an equation is to substitute the supposed value of the unknown quantity, and see if it satisfies the equation. Verification is a kind of proof of the correctness of the result obtained in such examples as these.*

Ex. 6—26. Solve the following equations, and verify on the values obtained:

6. $x+7+\frac{2x}{5}+3=24.$

7. $2x+\frac{2x}{3}+9=25.$

8. $2x+\frac{x}{2}-\frac{3x}{5}=7\frac{3}{5}.$

9. $\frac{x+1}{3}-16=\frac{x+4}{5}-\frac{x+4}{3}$. (See 89.)

10. $\frac{21-3x}{3}-\frac{2(2x+3)}{9}=6-\frac{5x+1}{4}$.

11. $x-\frac{4x+2}{5}=3x-1\frac{1}{3}$.

12. $\frac{3x+4}{5}-\frac{7x-3}{2}=\frac{x-16}{4}$.

13. $\frac{x}{2}-\frac{x}{3}=\frac{6(x+2)}{8}-5$.

14. $\frac{x+1}{3}+\frac{x+3}{4}=\frac{x+4}{5}+16$.

15. $\frac{4(x+2)}{3}-\frac{3x+1}{2}=1$.

---

16. Solve $2a^2x-x=a^2+a^3$. $x=\frac{a^2(1+a)}{2a^2-1}$.

17. Solve $a^2b+x=a^2c-cx+a^2$. $x=\frac{a^2(c-b+1)}{c+1}$.

18. Solve $ab+cx=bc+bx$.

19. Solve $\frac{a}{bx}+\frac{b}{cx}=d$. $x=\frac{ac+b^2}{bcd}$

20. Solve $3x-a=x-\frac{bx-d}{3}$.

21. Solve $ax-c=\frac{x-b}{a+c}$. $x=\frac{ac+c^2-b}{a^2+ac-1}$.

---

22. Solve $\frac{x^2}{5}+3x=7x-\frac{3x^2}{5}$.

Suggestion.—Observe that if each member is divided by x, there will be no term containing an x^2.

23. Solve $\frac{3x}{4}-\frac{2x}{5}=\frac{x^2-10x}{2}$. $x=10\frac{7}{10}$.

24. Solve $\frac{3}{5}x^2+\frac{1}{2}x=x+\frac{x^2+x}{4}$.

25. Solve $\frac{3x^2}{2}+5=\frac{2x-4}{7}+\frac{3x^2+1}{2}$.

Suggestion.—Observe that the terms containing x^2 destroy each other in the process.

26. Solve $\frac{a(c^2+x^2)}{cx}=ab+\frac{ax}{c}$. $x=\frac{c}{b}$.

SECTION XVII.

APPLICATIONS.

101. We saw in Section XIV. how practical problems are solved by means of the equation. Having, in Sections XV. and XVI., attempted to gain a better acquaintance with this wonderful instrument—this patent for solving problems—the equation, we shall now resume the study of the solution of problems.

102. The Algebraic Solution *of a problem consists of two parts:*

1st. **The Statement,** *which consists in expressing by one or more equations the conditions of the problem.*

2d. **The Solution** *of these equations so as to find the values of the unknown quantities in known ones. This process has been explained in the case of Simple Equations, in the preceding Sections.*

103. The *Statement* of a problem requires some knowledge of the subject about which the question is asked. Often it requires a great deal of this kind of knowledge in order to "state a problem." This is not Algebra; but it is knowledge which it is more or less important to have according to the nature of the subject.

104. Directions *to guide the Student in the Statement of Problems.*

1st. Study the meaning of the problem, so that, *if you had the answer given, you could prove it,* noticing carefully just what operations you would have to perform upon the answer in proving. This is called, *Discovering the relations between the quantities involved.*

2d. Represent the unknown (required) quantities (the answer) by some one or more of the final letters of the alphabet, as $x, y, z,$ or w, and the known quantities by the other letters, or, as given in the problem.

3d. Lastly, by combining the quantities involved, *both known and* *unknown*, according to the conditions given in the problem (as you would to prove it, if the answer were known) express these relations in the form of an equation.

1. Anna is three years younger than Eliza, and Eliza is seven years older than Lucy. The sum of their ages is seventeen. How old is each?

STATEMENT.—As the inquiry is concerning the ages of Anna, Eliza, and Lucy, we may represent either's age by x. For example if Anna's age is x, since Eliza is 3 years older than Anna, Eliza's age will be $x+3$; and as Eliza is 7 years older than Lucy, $x+3-7$ is Lucy's age. Thus we have

$$\begin{aligned} \text{Anna's age} &= x, \\ \text{Eliza's age} &= x+3, \\ \text{and Lucy's age} &= x+3-7. \end{aligned}$$

But the problem states that the sum of their ages is 17; hence

$$x+x+3+x+3-7=17,$$

is the equation.

SOLUTION.—With this the student is sufficiently familiar. x Anna's age, is found to be 6; $x+3$, Eliza's age, is 9; and $x+3-7$, Lucy's age, is 2.

2. If you give twenty-four cents more to James, he will have five times as many as he now has. How many has he?

3. What number must be added to itself and to seven more, that the sum may be three times that number?

Ans., 7.

4. When George shall be thirty years older, his age will be four times as much as it is now. What is his age?

5. A man can shingle one-fourth of the roof of a house in one day, and a boy can shingle one-twelfth of it in a day. How many days will it take for both, working together, to shingle the roof?

Suggestion.—Let x represent the required number of days. Then, as the man does $\frac{1}{4}$ of it in one day, he does x times $\frac{1}{4}$ or $\frac{x}{4}$, in x days. So the boy does $\frac{x}{12}$ in the time. But as they do once the work, $\frac{x}{4}+\frac{x}{12}=1$.

6. How many times the sum of one-third, one-seventh, and one-twenty-first, does it take to make a whole one?

Suggestion.—We have $x(\frac{1}{3}+\frac{1}{7}+\frac{1}{21})=1$, which may be put into the form $\frac{x}{3}+\frac{x}{7}+\frac{x}{21}=1$; or, better, $\frac{11x}{21}=1$.

7. One man can do one-third of a given piece of work in one day; another can do one-eighth of the same work in a day; and a boy can do one-twenty-fourth of it in the same time. How many days will it take the three, working together, to get it done?

8. A boy ate one-fourth of his plums, and gave away one-fifth of them. The difference between what he ate and what he gave away was three. How many had he? and how many did he give away? *Ans.*, He had 60.

9. If three-eighths of some number be taken from three-fourths of the same number, the remainder will be six. What is the number?

10. If from half of a man's money one-seventh of his

money be taken, the difference will be fifteen dollars. How many dollars has he?

11. The difference between three-fourths and five-sixths of the same number is nine. What is the number?

12. A man owned seven-tenths of a flock of sheep. After selling two-fifths of the whole flock, he had thirty sheep still belonging to him. How many sheep were in the flock before the sale? *Ans.*, 100.

13. Divide seventeen into two such parts, that twice one part shall be eight less than five times the other. What are the numbers?

14. A farm containing twenty-six acres, belongs to two men. Three times A's part is six acres less than four times B's part. How many acres has each?

15. Divide twenty-five into two such parts, that three times one part shall be three more than five times the other. What are the parts? *Ans.*, 16 and 9.

16. A boy, after spending a part of his money, found he had remaining three times as much as he had spent. He had twelve cents at first. How much did he spend? and how much was left?

17. A man had thirty-two sheep. After selling a part of his flock, he found the remainder was four less than twice the number he sold. How many did he sell? and how many were left?

18. Divide twenty-eight into two such parts, that, if one-fourth of the greater be taken from the whole num-

ber, the difference will be twice the less number. What will the parts be?

19. A cow and sheep cost thirty dollars. If the cost of the cow be taken from twice the cost of both, the remainder will be seven times the cost of the sheep. What was the cost of each?

Ans., Cow $25, Sheep $5.

20. Divide thirty-two into two such parts, that if four-fifths of the greater be taken from twice the whole number, the remainder will be four times the less number. What are the parts?

21. A man and boy received thirteen dollars for a week's labor. If two-thirds of what the man received be taken from twice the sum that both had, the difference will be five times the money which the boy received. How many dollars had each?

22. The garrison of a certain town consists of 125 men, partly cavalry and partly infantry. The monthly pay of a cavalry man is $20, and that of an infantry man is $15; and the whole garrison receives $2,050 a month. What is the number of cavalry, and what of infantry?

23. A gentleman is now 40 years old, and his son is 9 years old. In how many years, if they both live, will the father be only twice as old as his son?

24. A farmer wishes to mix rye worth 72 cents a bushel, with oats worth 45 cents a bushel, so that he may have

100 bushels worth 54 cents a bushel. How many bushel of each sort must he take?

Ans., $33\frac{1}{3}$ of rye, and $66\frac{2}{3}$ of oats.

25. In a mixture of copper, tin, and lead; 16 pounds less than $\frac{1}{2}$ was copper, 12 pounds less than $\frac{1}{3}$ was tin, and 4 pounds more than $\frac{1}{4}$ was lead. What was the weight of the whole mixture; and also of each kind?

Ans., 288lb.; and also 128lb., 84lb., and 76lb.

26. Of a piece of metal, $\frac{1}{3}$ plus 24 ounces is brass, and $\frac{3}{4}$ minus 42 ounces is copper. What is the weight of the piece?

27. Divide \$183 between two men, so that $\frac{4}{7}$ of what the first receives, shall be equal to $\frac{3}{10}$ of what the second receives. What will be the share of each?

Ans., \$63, and \$120.

28. A gentleman invested $\frac{3}{8}$ of his property in a canal. When he sold out, he lost $\frac{2}{3}$ of the sum invested, receiving only \$1,446. What was the value of his property when he began? *Ans.*, \$11,568.

29. A gentleman leaves \$315 to be divided among four servants in the following manner: B is to receive as much as A, and $\frac{1}{2}$ as much more; C is to receive as much as A and B, and $\frac{1}{3}$ as much more; D is to receive as much as the other three, and $\frac{1}{4}$ as much more. What is the share of each?

30. A man bought a horse and chaise for \$341. If $\frac{3}{8}$ of the price of the horse be subtracted from twice the

price of the chaise, the remainder will be the same as if $\frac{5}{7}$ of the price of the chaise be subtracted from three times the price of the horse. What was the price of each? *Ans.*, Chaise, \$189; horse, \$152.

31. Divide the number 204 into two such parts, that if $\frac{2}{5}$ of the less be subtracted from the greater, the remainder will be equal to $\frac{3}{7}$ of the greater subtracted from four times the less.

---

105. The following examples are exactly like the preceding except that the known or given quantities are represented by letters instead of figures. The corresponding example in the preceding set should be reviewed in connection with each of the following, and its answer deduced from the literal answer.

1. Anna is a years younger than Eliza, and Eliza is b years older than Lucy. The sum of their ages is c How old is each?

Suggestion.—See Ex. 1 above. As in that,

let x=Anna's age.
Then $x+a$ = Eliza's age,
and $x+a-b$ = Lucy's age.
Then $x+x+a+x+a-b=c$, is the equation.

Solving, $3x+2a-b=c$, or $3x=c+b-2a$,

$$\text{and } x = \frac{c+b-2a}{3}, \text{ Anna's age.}$$

$$\text{Hence } x+a=\frac{c+b-2a}{3}+a=\frac{c+b+a}{3}, \text{ Eliza's age.}$$

$$\text{and } x+a-b=\frac{c+b+a}{3}-b=\frac{c+a-2b}{3}, \text{ Lucy's age.}$$

Applying these results to Ex. 1 of the preceding set, $a=3$, $b=7$ and $c=17$.

Hence $\frac{c+b-2a}{3}=\frac{17+7-2\cdot3}{3}=6$, Anna's age.

$\frac{c+b+a}{3}=\frac{17+7+3}{3}=9$, Eliza's age.

and $\frac{c+a-2b}{3}=\frac{17+3-2\cdot7}{3}=2$, Lucy's age.

2. If you give a cents more to James, he will have b times as many as he now has. How many has he? Having obtained the answer in the literal notation, deduce that of Ex. 2, in the preceding set, by substituting the values there given.

Ans., $\frac{a}{b-1}$.

3. What number must be added to itself and a more, that the sum may be b times that number? Apply to Ex. 3 above.

4. Make an example like Ex. 4 of the preceding set, using a instead of the 30, and b instead of the 4. Why is the answer to this just like the answer to Ex. 2?

5. A man can shingle one ath of the roof of a house in one day, and a boy one bth of it in a day. How many days will it take for both, working together, to shingle the roof?

Suggestion.—One ath is expressed $\frac{1}{a}$. The equation is $\frac{x}{a}+\frac{x}{b}=1$. The answer is $\frac{ab}{a+b}$. Apply to Ex. 5 of the preceding set.

6. How many times one mth, one nth, and one rth, does it take to make a whole one?

Ans., $\frac{mnr}{mr+nr+mn}$.

7. Make and solve an example like Ex. 7 of the preceding set, using m, n, and r, for the numbers instead of 3, 8, and 24. How is this example different from the preceding?

8. Make and solve an example like Ex. 8 of the preceding, using a, b, and c, instead of 4, 5, and 3.

Ans., He had $\frac{abc}{b-a}$. He gave away $\frac{ac}{b-a}$.

9. If an $\frac{m}{n}$th part* of some number be taken from an $\frac{a}{b}$th part of the same number, the remainder will be d. What is the number?

Ans., $\frac{bdn}{an-bm}$.

10. Make and solve an example like the 10th of the preceding set, using letters instead of figures.

11. Make and solve literal examples like the 11th and 12th of the preceding set. Like which of the preceding are they? Why are these two alike?

13.† Divide a into two such parts, that m times one part shall be n times the other.

Parts, $\frac{am}{m+n}$, and $\frac{an}{m+n}$.

14. Make and solve literal examples like the 14th and 15th of the preceding set. Are these just alike?

* Read "an m divided by n th part."

† These numbers are made to correspond with those of the preceding set.

Make other examples like those of the preceding set, the answers to which can be found by substituting in the answer to this.

16. Make and solve literal examples like the 16th and 17th of the preceding set, using in the 17th, s instead of 32, a instead of 4, and n instead of 2. Are these examples exactly alike? If $a=0$ how will they differ?

106. Solutions *with letters instead of figures are called* **General Solutions,** *since they answer for all examples of the same kind.*

18. Make and solve *general* examples like those from the 18th to the 21st of the preceding, inclusive. In each case deduce from the literal, or *general* answer, the answer to the particular example.

Make other particular examples, whose answers can be deduced from these general answers.

22. The garrison of a certain town consists of s men, partly cavalry and partly infantry. The monthly pay of a cavalry man is $\$m$, and that of an infantry man is $\$n$; and the whole garrison receives $\$a$ a month. What is the number of cavalry, and what of infantry?

Ans., Cavalry men $\frac{a-ns}{m-n}$, infantry $\frac{ms-a}{m-n}$.

Query.—If you add the two answers together what ought they to amount to? Do they?

23. Make and solve *general* examples like the 23d and 24th of the preceding set, deducing the answers of those from the answers to these.

25. In a mixture of copper, tin, and lead, a pounds less than $\frac{1}{m}$* was copper, b pounds less than $\frac{1}{n}$ was tin, and c pounds more than $\frac{1}{r}$ was lead. What was the weight of the whole mixture; and also of each kind?

Ans., Of the whole $\dfrac{mnr(a+b-c)}{nr+mn+mr-mnr}$,

of copper $\dfrac{nr(a+b-c)}{nr+mn+mr-mnr}-a$,

of tin $\dfrac{mr(a+b-c)}{nr+mn+mr-mnr}-b$,

and of lead $\dfrac{mn(a+b-c)}{nr+mn+mr-mnr}+c$.

26. Of a piece of metal, $\frac{1}{m}$ part plus a ounces is brass, and $\frac{m}{n}$ part minus b ounces is copper. What is the weight of the piece.

Ans., $\dfrac{mn(b-a)}{n+m^2-mn}$.

27. Divide \$$a$ between two men, so that an $\frac{m}{n}$ part of what the first receives shall be equal to an $\frac{r}{s}$ part of what the second receives. What will be the share of each?

Ans., $\dfrac{anr}{ms+nr}$, and $\dfrac{ams}{ms+nr}$.

28. Generalize the 28th to the 31st of the preceding set, inclusive, and deduce from the general answers the answers to the particular examples.

* Read "one mth."

GEOMETRICAL
OR COMMON RATIO.
SECTION XVIII.

107. If we wish to compare 12 and 4 we can ask, "12 is how *much* greater than 4?" or, "12 is how *many times* as great as 4?"

The former inquiry is answered by giving the difference between 12 and 4, and the latter, by telling the quotient of 12, divided by 4.

Comparing two numbers by telling how *much* the former is greater than the latter, is sometimes called giving their *Arithmetical Ratio,* but more generally this method of comparison is spoken of simply as the difference.

Comparing two numbers by telling how many *times* as great the former is, as the latter, is called giving their *Geometrical Ratio,* or simply their *Ratio.*

The word *ratio* means *relation,* so that when we ask "What is the ratio of 12 to 4?" we ask, "What is the relation of 12 to 4?" As this is usually understood, we answer it by saying that 12 is three times as great as 4, or that the ratio of 12 to 4 is 3.

1. What is the ratio of 8 to 2? Of 10 to 5? Of 30 to 6? Of 128 to 4?

2. What is the ratio of mn to m? Of $3m^2r$ to m? Of $4m^2x^2$ to $2mx$? Of $15ax^3$ to $3x^2$?

Answer to last, $5ax$.

3. What is the ratio of 2 to 3? *Ans.,* $\frac{2}{3}$.

4. What is the ratio of 5 to 7? Of 3 to 5? Of 11 to 16? Of 117 to 13? Of 15 to 6? Of 24 to 66?

Answer to last, $\frac{4}{11}$.

5. What is the ratio of a to b?

Ans., $\frac{a}{b}$.

6. What is the ratio of $3m$ to $2n$? Of x to y? Of $11ma$ to $5cy$? Of $a-b$ to c? Of a^2-b^2 to $a-b$?

Answer to last, $a+b$.

7. What is the ratio of $3(a+b)^2$ to $2(a+b)$? To $3(m+n)$?

8. What is the ratio of $15(a^2+2ab+b^2)$ to $3(a+b)$?

Ans., $5(a+b)$.

108. Ratio *is the relative magnitude of one quantity as compared with another of the same kind, and is expressed by the quotient arising from dividing the first by the second. The first quantity named is called the* **Antecedent**, *and the second the* **Consequent.** *Taken together they are called the* **Terms** *of the ratio, or a* **Couplet.**

109. The Sign *of ratio is the colon,* :, *the common sign of division,* ÷, *or the fractional form of indicating division.*

Ill.—The ratio of 8 to 4 is expressed $8:4$, $8\div4$, or $\frac{8}{4}$, any one of which may be read "8 is to 4," or, "ratio of 8 to 4." The *antecedent* is 8, and the *consequent* 4. The sign : is an exact equivalent for ÷. The ratio of a to b is expressed $a:b$, $a\div b$, or $\frac{a}{b}$, each form meaning exactly the same thing.

9. Express the ratio of $3m$ to $5n$ in the three different forms?

10. Express the ratio of $a-b$ to $a+b$ in the three different forms?

11. Express the ratio of which $2n$ is the consequent, and $5m$ the antecedent. Also of which 7 is the antecedent, and $3x$ the consequent.

---

110. What effect upon the quotient is produced by multiplying the dividend?

By multiplying the divisor?

By dividing the dividend?

Dividing the divisor?

By multiplying or dividing both dividend and divisor by the same number?

111. Ask and answer the corresponding questions with regard to a fraction.

112. In a ratio what corresponds to dividend or numerator?

What to the divisor or denominator?

113. A ratio *being merely a fraction, or an unexecuted problem in Division, of which the antecedent is the numerator, or dividend, and the consequent the denominator, or divisor, any changes made upon the terms of a ratio produce the same effect upon its* *value, as the like changes do upon the value of a fraction, when made upon its corresponding terms. The principal of these are,*

1st. If both terms are multiplied, or both divided, by the same number, the value of the ratio is **not changed.**

2d. A ratio is **multiplied** *by multiplying the antecedent (i. e., the numerator or dividend), or by dividing the consequent (i. e., the denominator, or divisor).*

3d. A ratio is **divided** *by dividing the antecedent (i. e., the numerator, or dividend), or by multiplying the consequent (i. e., the denominator, or divisor).*

1. Multiply $a : b$ by m in two ways.

Results, $am : b$, and $a : \frac{b}{m}$.

2. Multiply $a+b : a^2 - b^2$ by $a + b$ in two way. Also by $a-b$.

3. Divide $3m^2y : 5nx$ by m in two ways.

4. Divide $(a+b)^2 : (a-b)^2$ by $a-b$ in two ways.

5. Reduce $15m^2y : 25my^2$ to its lowest terms.

Result, $3m : 5y$.

6. Reduce $12m^2 : 4my$ to its lowest terms.

---

7. If 8 is the antecedent and 2 the ratio, what is the consequent?

Having given the antecedent and the ratio, how do you find the consequent?

What question in division is the same as this?

8. If 6 is the consequent and the ratio is 5, what is the antecedent?

Having given the consequent and the ratio, how do you find the antecedent?

What question in division is the same as this?

9. What is the product of the consequent and ratio?

If $\frac{a}{b}=c$, what is a equal to?

10. Given r the ratio and a the antecedent, what is the consequent?

11. Given r the ratio and b the consequent, what is the antecedent?

---

12. Which is the greater $\frac{7}{11}$ or $\frac{8}{13}$?

How much?

How do you compare two fractions to ascertain which is the greater?

13. Which is the greater ratio 7 : 11, or 8 : 13?

Is there any difference between this question and the preceding?

14. Reduce the ratios 2 : 3, and 5 : 7 to ratios having a common consequent.

Reduce $\frac{2}{3}$ and $\frac{5}{7}$ to fractions having a common denominator.

15. Which is the greater 11 : 14, or 119 : 148 ?

---

16. What is the ratio of $\frac{3}{8}$ to $\frac{3}{4}$? Of $\frac{3}{5}$ to $\frac{4}{7}$? Of $\frac{a}{b}$ to $\frac{m}{n}$? Of $\frac{13m^2}{12n}$ to $\frac{26my}{6nx}$?

17. Let x be any quantity; what is the ratio of $3x$ to $4x$? Of $14x$ to $7x$?

18. Let y be any quantity; what is the ratio of $5y$ to $6y$? Of $7y$ to $8y$?

19. Let x be any number; what is the ratio of mx to nx? Of ax to bx?

20. In $3x : 5x$ does the value you assign to x affect the ratio?

If $x=1$, what is the ratio?

If $x=5$ what is the ratio? Then may $3x : 5x$ be considered as representing *any* two numbers having the ratio 3 : 5 ?

21. Represent *any* two numbers having the ratio 5 : 7.

Having the ratio 9 : 11.

Having the ratio m to n.

PROPORTION.

SECTION XIX.

114. **Proportion** *is an equality of ratios, the terms of the ratios being expressed. The equality is indicated by the ordinary sign of equality,* =, *or by the double colon* : : .

Thus, 8 : 4=6 : 3, or 8 : 4 : : 6 : 3, or 8÷4=6÷3, or $\frac{8}{4}=\frac{6}{3}$; all mean precisely the same thing. A proportion is usually read thus: "as 8 is to 4 so is 6 to 3."

NOTE.—The pupil should practice writing a proportion in the form $\frac{a}{b}=\frac{c}{d}$, still *reading* it "*a* is to *b* as *c* is to *d*." One form should be as familiar as the other. He must accustom himself to the thought that $a : b : : c : d$ means $\frac{a}{b}=\frac{c}{d}$ and *nothing more*. It will be seen that the language "8 is to 4 as 6 is to 3," means simply that $\frac{8}{4}=\frac{6}{3}$, for it is an abbreviated form for saying that "the relation which 8 bears to 4 is the same as (is equal to) that which 6 bears to 2;" that is, 8 is as many times 4 as 6 is times 3, or $\frac{8}{4}=\frac{6}{3}$.

115. **The Extremes** *(outside terms) of a proportion are the first and fourth terms. The* **Means** *(middle terms) are the second and third terms. Thus, in* **a : b=c : d**, **a** *and* **d** *are the* **extremes**, *and* **b** *and* **c** *are the* **means.**

1. Is 3 : 7 : : 6 : 11 a true proportion? Why? Is the fraction $\frac{3}{7}$ equal to $\frac{6}{11}$?

Which is the greater? How much?

2. Is $\frac{2}{3} : \frac{5}{7} :: 14 : 15$ a true proportion? Why? What is the ratio of $\frac{2}{3}$ to $\frac{5}{7}$?

3. Is $a : b :: ax : bx$ a true proportion? Why?

4. Is $\frac{2}{5} : 1\frac{1}{5} :: 3\frac{1}{7} : 9$ a true proportion? Why?

5. Is $\frac{2}{5} : 1\frac{1}{5} :: 3\frac{1}{7} : 9\frac{3}{7}$ a true proportion? Why?

Transformations.

116. As a proportion *is only an equation under a particular form, the same axioms (87) apply to its transformations as to the transformations of an equation.*

Thus $a : b :: c : d$ is the same as $\frac{a}{b} = \frac{c}{d}$. Hence the ratios correspond to the members of the equation, and

1st. *Any operation may be performed on either or both ratios which does not change the value of the ratio;*

2d. *Any operation may be performed on either or both ratios which changes both ratios alike.*

1. Can you multiply both antecedents by the same number without destroying the proportion? Why?

Try it by multiplying the antecedents of

$$12 : 6 :: 10 : 5,$$

by 3; thus $$36 : 6 :: 30 : 5.$$

Are both of these proportions true?

2. If $a : b :: c : d$, is $ma : b :: mc : d$? Why?

3. Can you multiply both terms of the same ratio by the same number without destroying the proportion? Why?

Try it by multiplying both terms of the first ratio in

$$3 : 15 :: 7 : 35, \text{ by } 6.$$

4. If $m : n :: x : y$, is $am : an :: x : y$? Why?
Have you performed an operation which does not change the value of the ratio, or one which changes both ratios alike?

5. If $m : n :: x : y$, is $\frac{m}{a} : n :: \frac{x}{a} : y$? Why?

Is $m : an :: x : ay$? Why?

Is $m : n :: \frac{x}{a} : \frac{y}{a}$? Why?

Is $m : n :: ax : by$? Why?

Is $am : n :: x : ay$? Why?

6. Try the questions in the preceding on the proportion $3 : 5 :: 21 : 35$, letting $a=4$ and $b=6$.

Suggestion.—To ascertain whether two ratios are equal, reduce them to ratios having a common consequent and compare their antecedents. (See Exs. 13–16, pages 130, 131.)

---

117. There is a very simple method by which we can discover all the transformations which can be made in a proportion without destroying it. Thus, if we know that

$$(1.) \qquad a : b :: c : d$$

is a true proportion, the ratio of a to b is the same as the ratio of c to d. Now let $\frac{a}{b} = r$, then $\frac{c}{d} = r$; whence $a = br$, and $c=dr$. Substituting these values of a and c, the proportion becomes

$$(2.) \qquad br : b :: dr : d.$$

Now, as (1) may represent *any* proportion, (2) also represents any proportion, and whatever transformations can be made on (2) without destroying the proportion can be performed on any proportion.

1. Is the product of the extremes of a proportion equal to the product of the means?

How do you see it from (2)?

2. Is the ratio of the antecedents of a proportion equal to the ratio of the consequents? *i. e.*, is $br : dr :: b : d$ a true proportion?

118. A proportion *is taken by* **Alternation** *when the means are made to change places, or the extremes.*

Thus $a : b :: c : d$ becomes by alternation either $a : c :: b : d$, or $d : b :: c : a$. The appositeness of the term *alternation* (taking every other one) is seen from the fact that the new order is obtained by taking the terms alternately; that is, 1st and 3d, 2d and 4th; or 4th and 2d, 3d and 1st.

3. If $a : b :: c : d$, is $d : b :: c : a$ a true proportion?

Suggestion.—If $a : b :: c : d$, we have $br : b :: dr : d$, as above Then we are to ascertain whether $d : b :: dr : br$. What is the value of the first ratio? Of the second?

4. If $a : b :: c : d$, does it follow that $a : d :: b : c$?

Suggestion.—Is $br : d :: b : dr$ necessarily true? What is the value of the first ratio? Of the second?

5. If $a : b :: c : d$, is $d : c :: b : a$?

NOTE.—All these inquiries are to be answered by examining the proportion in the form $br : b :: dr : d$.

6. If $a : b :: c : d$, is $b : a :: d : c$ a true proportion?

119. A proportion *is taken by* **Inversion** *when the terms of each ratio are written in inverse order.*

Thus, if $a : b :: c : d$, by inversion we have $b : a :: d : c$.

It is to be observed that in inversion the means are made extremes, and the extremes means.

7. Is $11 : 13 :: 44 : 52$ a true proportion?

Is it true if taken by inversion?

8. Write $a : b :: c : a$ first by inversion, and this result by alternation.

Is the last form a true proportion if the first is?

9. If you invert one of the ratios of a proportion, and do not the other, does it destroy the proportion; *i. e.*, is $b : br :: dr : d$ necessarily a true proportion?

---

10. If $a : b :: c : d$, does $a+b : a :: c+d : c$ follow?

Suggestion.—As above, r being the common ratio, the proportion $a : b :: c : d$ can be written $br : b :: dr : d$. Whence we have $br+b : br :: dr+d : dr$. What is the value of the first ratio? Of the second? Are they equal?

120. A proportion *is taken by* **Composition** *when the sum of the terms of each ratio is compared with either term of that ratio, the same order being observed in both ratios; or when the sum of the ante-*

cedents and the sum of the consequents are compared with either antecedent and its consequent.

Thus, if $a : b :: c : d$, by composition we have $a+b : a :: c+d : c$, or $a+b : b :: c+d : d$, or $a+c : b+d :: a : b$, or $a+c : b+d :: c : d$.

11. If $a : b :: c : d$, does $a+c : b+d :: a : b$ follow?
Does $a+c : b+d :: c : d$?
Does $a+d : b+c :: a : d$?

121. **If the difference** *instead of the* **sum** *be taken in the last definition, the proportion is taken by* **Division.**

12. If $a : b :: c : d$, does $a-c : a :: b-d : b$ follow?
Does $a-b : a :: c-d : c$?
Does $a-b : b :: c-d : d$?
Does $a+b : a-b :: c+d : c-d$?

Suggestion.—To examine the last form we put the proportion $a : b :: c : d$ in the usual form $br : b :: dr : d$, and then examine

$$br+b : br-b :: dr+d : dr-d.$$

Now the first ratio is $\frac{br+b}{br-d}$, or $\frac{r+1}{r-1}$; and the second is $\frac{dr+d}{dr-d}$, or $\frac{r+1}{r-1}$. Hence, the ratios being equal, the proportion is true.

---

13. If the first term of a proportion is unknown, and the other three known, how do you find the first? *i. e.*, solve $x : m :: a : n$ for x.

14. If the second term of a proportion is unknown, and the other three known, how do you find the second?

15. If the 4th term of a proportion is unknown, and the other three known, how do you find the fourth?

16. If $a : x :: x : b$, what is the value of x?

Suggestion.—Since the product of the extremes equals the product of the means $x^2 = ab$. Then extracting the square root of each member (Axiom 2, 87) we have $x = \sqrt{ab}$.

122. A Mean Proportional *between two quantities is a quantity to which either of the other two bears the same ratio that the mean does to the other of the two.*

Thus, if x is a mean proportional between a and b, a bears the same ratio to x that x does to b; *i. e.*, $a : x :: x : b$.

123. A Third Proportional *to two quantities is such a quantity that the first is to the second as the second is to this third (proportional).*

Thus, in the last proportion, b is a third proportional to a and x. So, also, a is a third proportional to b and x.

NOTE.—The pupil should notice carefully the language used in the last two definitions. We do not say "a mean proportional *to*," but "a mean proportional *between*," two others. So, again, we say "a third proportional *to* two others." Moreover, it is necessary that the two others be taken in the order named in the statement. Thus, if y is a third proportional to m and n, $m : n :: n : y$. But, if y is a third proportional to n and m, $n : m :: m : y$. Notice carefully the difference between the two statements.

17. Find a mean proportional between 4 and 9; *i. e.*, solve the proportion $4 : x :: x : 9$.

18. Find a mean proportional between $a-b$ and $a+b$.

19. Find a third proportional to 4 and 6. To 6 and 4.

Suggestion.—In one case we are to solve $4 : 6 :: 6 : x$, and in the other $6 : 4 :: 4 : x$.

20. Find a *Fourth Proportional* to 5, 7, and 6, in order, *i. e.*, solve the proportion $5 : 7 :: 6 : x$.

21. Find a fourth proportional to 5, 6, and 7, in order.

Find a fourth proportional to 7, 6, and 5, in order.

Having given three numbers, how many different fourth proportionals can be obtained?

124. A Fourth Proportional *to three numbers is the fourth term of a proportion of which the three are the 1st, 2d, and 3d in the order in which the numbers are named.*

Thus a fourth proportional to a, b, and c, would be x in the proportion $a : b :: c : x$. To b, c, and a it would be x in $b : c :: a : x$, etc.

RATIO AND PROPORTION.

PROBLEMS WHICH INVOLVE THEM.

SECTION XX.

1. Divide 36 into two parts which shall be to each other as 7 to 5.

Suggestion.—Let $7x$ and $5x$ be the parts. (See Ex. 17–21. Sec. xviii.)

2. Divide a into two parts which shall be to each other as m to n. *Parts*, $\frac{am}{m+n}$, $\frac{an}{m+n}$.

3. John and George had together 80 cents. George gave John 20 cents of his part, when they found their portions in the ratio of 2 to 3. How many cents had each at first?

Ans., John, 28; George, 52.

Suggestion.—The proportion is $x-20 : 100-x :: 2 : 3$, or by composition $x-20 : 80 :: 2 : 5$. Whence $5x-100=160$.

4. John and George had together a cents, George gave John b cents of his part, when they found their portions in the ratio of m to n. How many cents had each at first?

Ans., George, $\frac{am+bm+bn}{m+n}$; John, $\frac{an-bm-bn}{m+n}$.

5. Divide 30 into 3 parts, which shall be in the ratio of the numbers 2, 3, and 5.

6. Divide m into 3 parts, which shall be in the ratio of the numbers a, b, and c.

Parts, $\frac{am}{a+b+c}$, $\frac{bm}{a+b+c}$, and $\frac{cm}{a+b+c}$.

7. Four towns are situated in the order of the letters A, B, C, D. The distance from A to D is 120 miles; the distance from A to B is to the distance from B to C as 3 to 5; and one-third of the distance from A to B, added to the distance from B to C, is three times the distance from C to D? How far are the towns apart?

Ans., A to B, 36 miles; B to C, 60 miles; C to D 24 miles.

8. Four places are situated in the order of the 4 letters, A, B, C, and D; the distance from A to D is 34

miles; the distance from A to B is to the distance from C to D, as 2 to 3; and $\frac{1}{4}$ the distance from A to B, added to $\frac{1}{3}$ the distance from C to D, is 3 times the distance from B to C. Required the respective distances.
Distances, A to B, 12; B to C, 4; and C to D, 18 miles.

9. Divide the number 50 into two such parts that the greater increased by 5, may be to the less diminished by 5, as 7 to 3.

10. A footman started from a certain place, and traveled 4 miles an hour. After he had been gone 3 hours, a horseman started in pursuit, riding 7 miles an hour. How long before the horseman would overtake the footman? How far from the starting-place would the footman be overtaken?

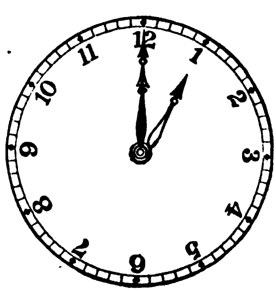
Suggestion.—How many miles would the horseman *gain* on the footman each hour? How many miles had he to gain before overtaking him. The proportion is

$$3 : 12 :: 1 : x,$$

x being the time required to overtake the footman.

11. The hour and minute hands of a clock are exactly together at 12 M. When are they next together?

Suggestion.—Measuring the distance around the dial by the hour spaces, the whole distance around is 12 spaces. Now, when the hour hand gets to 1, the minute hand has gone clear around, or over 12 spaces. But as the hour hand has gone *one* space, the minute hand has *gained* only 11 spaces. Now as the minute hand must *gain* an entire round, or 12



spaces, to overtake the hour hand, we have the question: If the minute hand gains 11 spaces in 1 hour, how long will it take to gain 12 spaces? $\therefore$ 11 : 12 : : 1 hour : x hours; and $x=1\frac{1}{11}$ hours, or 1 hour $5\frac{5}{11}$ minutes.

12. At what time between 3 and 4 o'clock does the minute hand pass the hour hand?

Suggestion.—Reckoning from 12 o'clock, when the hands are together, how many spaces must the minute hand gain in order to pass the hour hand between 3 and 4 o'clock? How many does it gain in an hour? It passes at 3 o'clock $16\frac{4}{11}$ minutes.

13. A's age is to B's as 4 to 3, and if twice B's age be added to A's, the sum will be 100 years. Required the age of each.

14. A ship and a boat are descending a river at the same time; and when the ship is opposite a certain fort, the boat is 13 miles ahead. The ship is sailing at the rate of 5 miles, while the boat is going 3. At what distance below the fort will they be together?

Ans., $32\frac{1}{2}$ miles.

15. A certain man found when he married, that his age was to that of his wife as 7 to 5. If they had been married 8 years sooner, his age would have been to hers as 3 to 2. What were their ages at the time of their marriage?

Ans., His age, 56 years; hers, 40.

16. A field of 864 acres is to be divided among three farmers, A, B, and C; so that A's part shall be to B's as

5 to 11, and C may receive as much as A and B together. How much must each receive?

Ans., A, 135; B, 297; C, 432 acres.

Simple Equations

WITH TWO UNKNOWN QUANTITIES.

SECTION XXI.

125. It frequently happens that a problem requires us to find several quantities, so that there is more than one unknown quantity. In such cases it is sometimes best to use two or more letters, each representing one of the quantities sought. Let us study such an example.

1. John and George's ages together amount to 29 years, and 3 times George's plus 5 times John's age is 113 years. What is the age of each?

SOLUTION.—We might solve this by letting x represent George's age, and $29-x$ John's, as we have done several in the preceding section. But we wish to do it in another way, in order to learn how to proceed with *two unknown quantities*. We therefore let

x represent George's age,
and y represent John's age.

Now the sum of their ages is 29; hence

$$x+y=29. \qquad \text{1st equation.}$$

Again the example says that 3 times George's age, *i. e.* $3x$, plus 5 times John's, *i. e.* $5y$, is 113;

$$\text{hence } 3x+5y=113. \qquad \text{2d equation.}$$

This is the *statement*.

In order to solve the equations let us find the value of y in each. Thus from the 1st, $y = 29 - x$.

From the 2d, $y = \frac{113 - 3x}{5}$.

Now as y means the same thing in both equations, $29-x$ is equal to $\frac{113-3x}{5}$, for each is the value of y. Hence we have

$$\frac{113-3x}{5}=29-x.$$

This being an equation having only one unknown quantity, we can find the value of x in known quantities. Solving, we have

$$113-3x=145-5x,$$
$$5x-3x=145-113,$$
$$2x=32,$$
$$x=16, \text{ George's age.}$$

Finally substituting this value of x for x in the equation $y=29-x$ we have

$$y = 29 - 16,$$
$$y = 13, \text{ John's age.}$$

126. We notice that there were *two* unknown quantities involved in this problem, and that there were *two* statements, each of which gave rise to an equation. *But the equations were not the same, although x represented the same thing in each, and also y.*

127. Independent Equations *are such as express* **different** *conditions, and neither can be reduced to the other.*

128. Simultaneous Equations *are those which express different conditions of the same problem, and consequently the letters representing the unknown quantities signify the same things in each. The equations of such a group are all satisfied by the same values of the unknown quantities.*

Queries.—Are the two equations which we used in the preceding example *independent?* Why? Can you transform $x+y=29$ into $3x+5y=113$?

Are these two equations simultaneous? Why?

If we took the equation $3x+5y=113$ and then made an equation $3y-2x=8$, from Ex. 15, in the preceding section, in which x represents the wife's age and y the husband's, would these two equations be independent? Why? Would they be simultaneous? Why?

Are $x+y=29$, and $2x+2y=58$, independent? Why? Can you make either form the other? How?

NOTE.—We can readily tell whether two equations are independent, but we can not tell whether they are simultaneous unless we know what each one means. Thus, were I to give you the two equations $x+y=29$, and $3y-2x=8$, you could not tell whether they were simultaneous or not, unless I told you what x and y meant in each. You could treat any two equations with two unknown quantities as simultaneous, if you chose, and find values for x and y which would satisfy them both. But if one of the equations had reference to one problem, and the other to a different problem, the values of x and y thus obtained would not refer to either problem.

---

129. We notice farther that in solving the last example we combined the two equations having two unknown quantities, so as to make one equation having but one unknown quantity.

130. Elimination *is the process of producing from a given set of simultaneous equations containing two or more unknown quantities, a new set of equations in which one, at least, of the unknown quantities shall not appear. The quantity thus disappearing is said to be eliminated.*

(The word literally means *putting out of doors.* We use it as meaning *causing to disappear.*)

181. *There are* **Three Methods of Elimination** *in most common use, viz., by* **Comparison,** *by* **Substitution,** *and by* **Addition or Subtraction.**

NOTE.—Any one of these methods will solve all problems; but some problems are more readily worked by one method than by another, while it is often convenient to use several of the methods in the same problem, especially when there are more than two unknown quantities. The method given above was comparison.

Elimination by Comparison.

If we examine the solution of the last example we shall see that it suggests the following

182. **RULE.—1st.** *Find expressions for the value of the same unknown quantity from each equation, in terms of the other unknown quantity and known quantities.*

2d. *Place these two values equal to each other, and the result will be the equation sought.*

DEM.—The first operations being performed according to the rules for simple equations with one unknown quantity, need no further demonstration.

2d. Having found expressions for the value of the *same* unknown quantity in both equations, since the equations are simultaneous this unknown quantity means the same thing in the two equations, and hence the two expressions for its value are equal.

NOTE.—The resulting equation can be solved by the rules already given.

1. Given $3x+2y=26,$
and $5x-2y=38,$
to find the values of x and y, eliminating by comparison.

Suggestion.—From the 1st, $y = \frac{26-3x}{2}$,

and from the 2d, $y = \frac{5x-38}{2}$.

Hence, since y means the same thing in both equations, *i. e.*, since the equations are simultaneous,

$$\frac{26-3x}{2} = \frac{5x-38}{2}.$$

Then,
$$26-3x = 5x-38,$$
$$8x = 64,$$
$$x = 8.$$

Substituting in
$$y = \frac{26-3x}{2},$$
$$y = \frac{26-24}{2} = 1.$$

NOTE.—In the following we shall assume that the equations are simultaneous.

2. Given $\begin{cases} 4y+\ \ x=102 \\ \ \ y+4x=\ \ 48 \end{cases}$ to find y and x.

$y=24$, $x=6$.

3. Given $\begin{cases} 2x-\ \ y=6 \\ 4x+3y=22 \end{cases}$ to find x and y.

$x=4$, $y=2$.

4. Given $\begin{cases} 4x+6y=46 \\ 5x-2y=10 \end{cases}$ to find x and y.

$x=4$, $y=5$.

133. NOTE.—In such examples the values of the unknown quantities as found *must satisfy both equations.* Thus, in the last example, substituting in the first equation, we have $4 \cdot 4+6 \cdot 5=46$, which is true. Also in the second, $5 \cdot 4-2 \cdot 5=10$, a true equation.

But we can readily find numbers which will satisfy one equation and not the other. Thus $x=10$ and $y=1$ satisfy the 1st of these equations, but not the second. So $x=6$, $y=10$, satisfies the second but not the first. *The true values must satisfy both equations.*

Solve and *verify* the following:

5. $3x=11+2y$, $7y-2x=21$.

6. $5y=128-6x$, $3x=88-4y$.

7. $2my=\dfrac{3x-n}{a}$, $mx-y=3x+a$.

Suggestion.—From the first, $y=\dfrac{3x-n}{2am}$; from the second, $y=(m-3)x-a$. Hence, $\dfrac{3x-n}{2am}=(m-3)x-a$. Solving this, $x=\dfrac{2a^2m-n}{2am^2-6am-3}$.

To obtain the value of y it is sometimes better to return to the original equations, and eliminate x as y was eliminated. Thus, from the first, $x=\dfrac{2amy+n}{3}$; from the second, $x=\dfrac{y+a}{m-3}$. Hence, $\dfrac{2amy+n}{3}=\dfrac{y+a}{m-3}$. Solving this, $y=\dfrac{3a+3n-mn}{2am^2-6am-3}$.

8. $x+y=a$, and $x-y=b$.

9. $\dfrac{a}{b+y}=\dfrac{b}{3a+x}$, and $ax+2by=c$.

$$x=\frac{2b^2-6a^2+c}{3a},\quad y=\frac{3a^2-b^2+c}{3b}.$$

Elimination by Substitution.

1. Given $\dfrac{x}{7}+7y=99$, and $\dfrac{y}{7}+7x=51$ to find the values of x and y by substitution.

SOLUTION.—From the 1st equation we find $x=693-49y$. Hence, as x means the same thing in the two equations, *i. e.*, the equations are simultaneous, we can substitute this value of x in the second equation; whence we have

$$\frac{y}{7}+7(693-49y)=51.$$

Solving this, $y=14$. Substituting this value of y for y in $x=693-49y$, we have $x=7$.

This solution suggests the following

184. RULE.—1st. *Find from one of the equations the value of the unknown quantity to be eliminated, in terms of the other unknown quantity and known quantities.*

2d. *Substitute this value for the same unknown quantity in the other equation.*

DEM.—The first process consists in the solution of a simple equation, and is demonstrated in the same way.

The second process is self-evident, since, the equations being simultaneous, the letters mean the same thing in both, and it does not destroy the equality of the members to replace any quantity by its equal.

Solve the following by substitution, and verify the results obtained.

1. $7x+3y=29$, and $5x+2y=20$.

2. $4x-7y=34$, and $8x=102-3y$.

3. $12x+3y=5y+36$, and $\frac{4x}{5}-\frac{2y-3}{7}=4y-\frac{8x+15}{5}-36$.

Suggestion.—First reduce the equations to simple forms by clearing of fractions and uniting terms. They become $6x=y+18$, and $25y-14x=230$.

4. $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$.

5. $x+y=60$, and $\frac{x-31}{31-y}=\frac{6}{7}$.

6. $2ax=x-3by+m$, and $ax+by=c$.

$$x=\frac{3c-m}{a+1},\ y=\frac{a(m-2c)+c}{b(a+1)}.$$

7. $\frac{x}{a}+\frac{y}{b}=2$, and $bx-ay=0$. $x=a,\ y=b$.

8. $2x+\frac{y-2}{5}=21$, and $4y+\frac{x-4}{6}=29$.

9. $\frac{1-3x}{9}+\frac{3y-1}{5}=2$, and $\frac{3x+y}{11}+y=9$.

Elimination by Addition or Subtraction.

1. Given $\frac{5y}{4}-10+x=10+\frac{13+x}{4}$, and $\frac{y}{10}+\frac{x}{4}-1$ $=3+\frac{x-y}{20}$.

Solution.—Reducing the equations to the simplest form, we have

(1) $5y+3x=93$,

and (2) $3y+4x=80$.

To eliminate y, first multiply both members of the first equation by 3, and of the second, by 5. There results,

(3) $15y+\ 9x=279$,

(4) $15y+20x=400$.

Now y has the same co-efficient in both equations. Hence, if we subtract $15y+9x$ from $15y+20x$, we have $11x$ left in the 1st member of (4). But we must subtract just as much from the 2d member, or else we shall not have a true equation. Hence, as 279 is just the same as $15y+9x$, we will subtract it from 400; whence 121 is the second member, and we have

$$11x=121\text{ ; or } x=11.$$

This operation will be seen more clearly if we write (3) under (4), and then subtract member from member, thus,

$$\begin{array}{ll} (4) & 15y+20x=400, \\ (3) & \underline{15y+\ \ 9x=279,} \\ & \qquad 11x=121,\ \text{whence } x=11. \end{array}$$

To find the value of y, we may eliminate x by subtraction. To do this, multiply the members of (1) by 4, and of (2) by 3, whence,

$$\begin{array}{r} 20y+12x=372, \\ \underline{9y+12x=240,} \end{array}$$

Subtract and $\qquad 11y=132$, whence $y=12$.

Having found the value of x, we could have substituted in (1), or in (2), as we have done before.

2. Given $2x+3y=7$, and $8x-10y=6$, to eliminate y by addition.

Suggestion.—Multiplying both members of the 1st by 10, and of the 2d by 3, we have

$$20x+30y=70,$$

and $\qquad \underline{24x-30y=18.}$

Now, adding the corresponding members, we have

$$44x=88,\ \text{whence } x=2.$$

This value of x may be substituted in either of the equations, and the value of y determined. But we will eliminate x in the same manner as we did y. Thus, multiplying the members of the 1st equation by 4, we have

$$8x+12y=28,$$

The 2d equation is $\qquad \underline{8x-10y=\ \ 6.}$

Subtracting, $\qquad 22y=22$, whence $y=1$.

Queries.—What did we subtract from the first member of $8x+12y=28$? What from the second? From which member did we subtract the most? If two things are equal, and we take the same amount from each, how will the remainders compare?

From these two solutions the following rule *for elimination by addition or subtraction* will be naturally suggested.

185. RULE.—1st. *Reduce the equations to the forms* $ax+by=m$, *and* $cx+dy=n$.

2d. *If the co-efficients of the quantity to be eliminated are not alike in both equations, make them so by finding their L. C. M., and then multiplying each equation by this L. C. M. exclusive of the factor which the term to be eliminated already contains.*

3d. *If the signs of the terms containing the quantity to be eliminated are alike in both equations, subtract one equation from the other, member by member. If these signs are unlike, add the equations.*

DEM.—The first operations are performed according to the rules already given for clearing of fractions, transposition, and uniting terms, and hence do not vitiate the equations. The object of this reduction is to make the two subsequent steps practicable.

The second step does not vitiate the equations, since in the case of either equation, both its members are multiplied by the same number.

The 3d step eliminates the unknown quantity, since, as the terms containing the quantity to be eliminated have the same numerical value, if they have the *same* sign, by *subtracting* the equations one will destroy the other, and if they have different signs, by *adding* the equations they will destroy each other. The result is a true equation, since, if equals (the two members of one equation) are added to equals (the two members of the other equation), the sums are equal. Thus we have a new equation with but one unknown quantity.

3. Given $2x+3z=38$, and $6x+5z=82$ to find x and z, eliminating by addition or subtraction.

Suggestion.—The co-efficient of x in the first equation can be made equal to that in the second by multiplying by 3, hence we will eliminate x. It would take more work to eliminate z. Why?

$$\begin{aligned} 6x+9z&=114, \\ 6x+5z&=\ 82, \\ \hline 4z&=\ 32. \\ z&=\ \ 8. \end{aligned}$$

It is customary when we use addition or subtraction to eliminate one of two unknown quantities, to find the other by substitution. Thus substituting 8 for z, in $2x+3z=88$, we find $x=7$.

In solving the following, eliminate by addition or subtraction first, and having found the value of one unknown quantity, find the other by substituting in the *simplest* of the given equations.

4. $x+2y=17$, and $3x-y=2$. $\qquad x=3,\ y=7.$

5. $\dfrac{7x}{2}+2y=29$, and $3x=12\frac{2}{3}-\dfrac{4y}{3}$. Verify.

6. $\frac{1}{3}x-\frac{1}{4}y=2$, and $\frac{1}{4}x+\frac{1}{2}y=7$. Verify.

7. $\dfrac{2x}{3}-\dfrac{5-y}{2}=\dfrac{41}{12}-\dfrac{2x-1}{4}$, and $x+1:y::5:3$.

$x=4,\ y=3.$

8. $x+y:4x+y::4:7$, and $\dfrac{11y}{30}-\dfrac{2x}{5}+\dfrac{21-3y}{4}=\dfrac{2}{3}$ $+\dfrac{x}{30}-\dfrac{1}{6}$. $\qquad x=3,\ y=9.$

9. $\frac{a}{x}+\frac{b}{y}=m$, and $\frac{c}{x}+\frac{d}{y}=n$.

Suggestion.—It is not always best to reduce the equations to the simplest form. In this example, if we multiply the members of the 1st by c, and of the 2d by a, we can eliminate x by subtraction. Thus,

$$\frac{ac}{x}+\frac{bc}{y}=cm,$$

$$\frac{ac}{x}+\frac{ad}{y}=an.$$

Subtracting, $$\frac{bc-ad}{y}=cm-an.$$

Taking the reciprocals of each member (why does this give a true equation?),

$$\frac{y}{bc-ad}=\frac{1}{cm-an}, \text{ or } y=\frac{bc-ad}{cm-an}.$$

Finding the value of x in a similar manner, we have

$$\frac{ad}{x}+\frac{bd}{y}=dm,$$

$$\frac{bc}{x}+\frac{bd}{y}=bn.$$

$$\frac{ad-bc}{x}=dm-bn.$$

$$\frac{x}{ad-bc}=\frac{1}{dm-bn}.$$

Whence, $$x=\frac{ad-bc}{dm-bn}.$$

10. $\frac{4}{x}=\frac{4}{y}-1$, and $\frac{2}{x}-\frac{4}{y}=-\frac{3}{2}$.

Solve the following by such of the above methods as are most convenient:

11. $7x+3y=29$, and $5x+2y=20$.

12. $2x=3y$, and $5x-y=72$.

13. $y=\frac{2x-1}{3}$, and $5y=4x-4$. $x=3\frac{1}{2}$, $y=2$.

14. $\frac{1}{x}+\frac{1}{y}=m$, and $\frac{1}{x}-\frac{1}{y}=n$.

$$x=\frac{2}{m+n},\ y=\frac{2}{m-n}.$$

15. $x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2}$, $y+\frac{y-3}{x-18}=30-\frac{73-3y}{3}$.

$x=21$, $y=20$.

SECTION XXII.

PROBLEMS GIVING RISE TO TWO EQUATIONS EACH.

1. A can perform a piece of work in 20 days, B and C can together do it in 12 days. Now, if they all work for 6 days, C can finish it in 3 days. In what time would B or C have done it alone?

Suggestion.—Let x days be the time C would require to do it alone, and y days the time B would require. Then C does $\frac{1}{x}$ in a day, and B $\frac{1}{y}$. In 12 days, therefore, C and B would do $\frac{12}{x}+\frac{12}{y}=1$, the whole work.

In 6 days, all working together do $\frac{6}{20}+\frac{6}{x}+\frac{6}{y}$. To this add what C would do in 3 days, viz., $\frac{3}{x}$, and the whole would be done.

Hence, $\frac{6}{20}+\frac{6}{x}+\frac{6}{y}+\frac{3}{x}=1$.

Reducing the last equation, the two become,

$$\frac{12}{x}+\frac{12}{y}=1,$$

and $$\frac{9}{x}+\frac{6}{y}=\frac{7}{10}.\quad \text{Therefore, } x=15, y=60.$$

To solve these, see Ex. 9, page 154.

2. It is required to find a fraction such, that if 3 be subtracted from the numerator and denominator, it is changed into $\frac{1}{4}$, and if 5 be added to the numerator and denominator, it becomes $\frac{1}{2}$. What is the fraction?

Ans., $\frac{7}{19}$.

3. A can do a piece of work in 20 days, and B and C can together perform it in 12 days. Now, if all three work for 6 days, C can finish it in 3 days. It what time would B or C have performed it?

Ans., B 60 days, C 15 days.

4. Charles bought five peaches and two pears for seventeen cents, and found that two pears cost four cents less than two peaches. What did one of each cost?

5. A farmer bought three sheep and a cow for twenty-six dollars. At the same rate, a cow would cost four dollars less than twelve sheep. What did he pay for the cow, and what for a sheep?

6. A man bought a cow and ten sheep for forty dollars. He then sold, at the same rate, seven sheep and a cow for thirty-four dollars. What was the price of one of each?

7. If three times Anna's age be added to three times

Mary's age, the sum will be thirty-three years; and three times Mary's age is thirty-seven years less than seven times Anna's. What are their respective ages?

8. A certain number, consisting of two places of figures, is equal to seven times the sum of its digits, and if 18 be subtracted from it, the digits will be inverted. What is the number?

Suggestion.—Let x represent the *tens* figure, and y the *units.* Then $10x+y$ represents the number. The equations are $10x+y=7(x+y)$, and $10x+y-18=10y+x$. The number is 42.

9. A certain number consists of two digits, and is equal to the difference of the squares of its digits. If 36 be added to it, the sum will be expressed by the same digits in an inverted order. What is the number?

Ans., 48.

10. There are two numbers in the ratio of 5 : 4, but if each be increased by 20, the results are as 9 : 8. What are the numbers? *Ans.*, 25 and 20.

11. What fraction is that to the numerator of which if 1 be added, the fraction will equal $\frac{1}{3}$, but if 4 be added to the denominator, the fraction becomes $\frac{1}{4}$?

Suggestion.—Let x be the numerator and y the denominator, so that the fraction is $\frac{x}{y}$. The equations are $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+4}=\frac{1}{4}$. The fraction is $\frac{7}{24}$.

12. What fraction is that which becomes $\frac{1}{4}$ when 1 is added to both numerator and denominator, and 5 when

the numerator is multiplied by 25, and the denominator diminished by 1?

13. Bought linen at 60 cts. per yard, and muslin at 15 cts. per yard, amounting in all to $11.40. I afterwards sold $\frac{1}{3}$ of the linen and $\frac{1}{4}$ of the muslin for $3.89, having made 29 cents on this part. How many yards of each did I purchase?

Ans., 15 linen, 16 muslin.

Queries.—If there were x yds. of linen, and y of muslin, what was the cost? What was the cost of what he sold, in terms of x and y?

14. Purchased 25 lbs. of sugar, and 36 of coffee, for $8.04, but the price of each having fallen 1 cent per pound, I afterwards bought 2 lbs. more of the first, and 3 lbs. more of the second, for the same money. What was the price of each?

15. Two men in partnership divide their gain, so that the sum of twice A's share, added to B's share, makes twenty-seven dollars; and if three times B's money be taken from four times A's, nineteen dollars will be left. How many dollars will each have?

16. A man said, that if one-half the price of his saddle were taken from one-fifth of the price of his horse, the difference would be fifteen dollars; but one-tenth of the price of his horse and one-tenth of the price of his saddle together would be eleven dollars. What was the price of each?

17. The sum of two-thirds of the greater of two num-

bers added to the less is twelve? but the sum of one-fourth of both is only four. What are the numbers?

18. A man said that the sum of four-fifths of the value of his horse, added to two-thirds of the value of his cart, was forty-two dollars, and that the difference between one-third of the value of his cart and three-fifths of the value of his horse, was nineteen dollars. What was the value of each?

19. Four pounds of coffee and three pounds of tea cost \$4.98; but when coffee fell $16\frac{2}{3}\%$, and tea rose 20%, six pounds of coffee and three pounds of tea cost \$6.00. What was the price before the rise?

Ans., Coffee, 12 cts.; tea, \$1.50.

20. A man bought coffee at 12 cents, and tea at 75 cents a pound, and paid for the whole \$249; the next day he disposed of $\frac{1}{3}$ of his coffee and $\frac{4}{5}$ of his tea for \$180, which was \$10.80 more than it cost him. How many pounds of each article did he buy, and how much of each did he sell?

Simple Equations

WITH THREE UNKNOWN QUANTITIES.

SECTION XXIII.

1. Given $\left\{\begin{array}{l} x+y+z=15 \\ x+2y+3z=23 \\ x+3y+4z=28 \end{array}\right\}$ to find the values of x, y, and z.

SOLUTION.—Subtracting the members of the 1st from those of the 2d,

$$y+2z=8.$$

In like manner, from 2d and 3d, $y+z=5$. We now have

$$y+2z=8,$$

and

$$y+\ z=5,$$

two equations with *two* unknown quantities. Solving these as already learned, we find $z=3$, and $y=2$. These values substituted in any one of the given equations, give $x=10$.

From this example we readily infer the following rule for elimination, when there are three equations with three unknown quantities:

136. RULE.—*Combine one of the equations with each of the other two, so as to eliminate the same unknown quantity from each. There will thus result two equations with two unknown quantities.*

These can be solved by the methods of the preceding sections.

2. Given $\left\{\begin{array}{r} 7x+5y+2z=79 \\ 8x+7y+9z=122 \\ x+4y+5z=55 \end{array}\right\}$ to find the values of x, y and z.

Ans., $x=4$, $y=9$, and $z=3$.

3. Given $\left\{\begin{array}{r} 4x-3y+2z=10 \\ 5x+6y-8z=-1 \\ -x+8y+3z=44 \end{array}\right\}$ to find the values of x, y, and z.

Ans., $x=3$, $y=4$, and $z=5$.

4. Given $\left\{\begin{array}{l} 5x+3y+2z=29 \\ 2x+5y-z\ =14 \\ 3x-2y+4z=20 \end{array}\right\}$ to find the value of x, y, and z.

5. Given $\left\{\begin{array}{l} \dfrac{x+y}{3}+2z=21 \\ \dfrac{y+z}{2}-3x=-65 \\ \dfrac{3x+y-z}{2}=38 \end{array}\right\}$ to find the values of x, y, and z.

Ans., $x=24$, $y=9$, and $z=5$.

6. Given $\left\{\begin{array}{l} \frac{2}{3}x-\frac{1}{2}y+\frac{3}{4}z=4 \\ \frac{1}{2}x+\frac{2}{3}y-3z=-13 \\ 3x-2y+z=2 \end{array}\right\}$ to find the value of x, y, and z.

Ans., $x=6$, $y=12$, and $z=8$.

Problems.

1. James, Henry, and George, have each a certain number of cents.

If James gives Henry 5 and George 3, he will then have $\frac{2}{3}$ as many as both of them; but if George had 20 more than he has, he would have half as many as the other two. George's money, plus $\frac{3}{4}$ of Henry's, equals James's. How much has each?

Ans., James, 60; Henry, 40; George, 30.

2. What number is that expressed by three digits, to which if you add 297 the order of the digits will be reversed; the number expressed by the last two of which is twice that expressed by the first two, less 3; and 3 times the difference between the extreme digits is 1 more than twice the mean. What is the number?

Suggestion.—Letting x represent the hundreds digit, y the tens, and z the units, the number is represented by $100x+10y+z$.

The equations are

$$100x+10y+z+297=100z+10y+x.$$

$$10y+z=2(10x+y)-3.$$

and $$3(z-x)=2y+1.$$ The number is 245.

3. A merchant bought at one time 4 barrels of flour, 3 barrels of rice, and 2 boxes of sugar for $72; at another, 2 barrels of flour, 5 barrels of rice, and 3 boxes of sugar for $84; and at a third time, 5 barrels of flour, 9 barrels of rice, and 8 boxes of sugar for $187. What were the flour and rice per barrel, and what was the sugar per box?

4. Three boys, A, B, and C, counting their money, it was found that twice A's added to B's and C's, would make $5.25; that if A's and twice B's were added, and from the sum C's were subtracted, the result would be $3.00; and the three together had $3.25. How much money had each?

5. Three men owed together a debt of $1000, but neither of them had sufficient money to pay the whole alone. The first could pay the whole, if the second and third would give him $\frac{5}{14}$ of what they had; the second could pay it, if the first and third would give him $\frac{4}{13}$ of what they had; and the third could pay it if the first and second would give him $\frac{8}{11}$ of what they had. How much money had each?

SECTION XXIV.

MEANING OF FRACTIONAL AND NEGATIVE EXPONENTS.

In SECTION II. we learned that a figure written at the right and a little above a letter or figure, is *one form* of what is called an EXPONENT, and it was there promised that we should learn more upon this matter. We will first attend to some definitions which, though they have been learned in arithmetic, need to be made very familiar.

187. A Power *of a number is the product which arises from multiplying the number by itself, i. e., taking it a certain number of times as a factor.*

ILLUSTRATION. 8 is a *power* of 2 because it is the product arising from multiplying 2 by itself. So also 16, 32, 64, 128, etc., are *powers* of 2. Again, a^2, a^3, a^4, a^5, etc., are *powers* of a, since they are products arising from multiplying a by itself. Is 27 a power of 3? Why? How many times is 3 taken as a factor in 27? Is x^5 a power of x? Why? How many times is x taken as a factor in x^5? Is 12 a power of 2? Of 3? Why not? Is ab a power of a, or of b? Can you make ab by taking either a or b only, as a factor?

---

188. A Root *of a number is one of several* **equal** *factors into which the number is to be resolved.*

ILLUSTRATION. 2 is a root of 8? Why? Is 3 a root of 27? Of 81? Of 7? Of 12? Is a a root of a^2? Of a^3? Of a^5? Why? Is a a root of ab? Of cx? Why not?

189. The Square or Second Root *means one of the two equal factors of a number, and is indicated by the radical sign,* $\sqrt{\ }$, *or by the fractional exponent* $\frac{1}{2}$.

The Cube or Third Root *means one of the three equal factors of a number, and is indicated by the radical sign with 3 in the opening thus* $\sqrt[3]{\ }$, *or by the fractional exponent* $\frac{1}{3}$.

Fourth, Fifth, *and higher roots have similar meanings and are indicated in a similar manner.*

1. Read $\sqrt{4}$*.
What does it mean?
What is its value?

2. Read $4^{\frac{1}{2}}$. (This is read "4, exponent $\frac{1}{2}$.")
What does it mean?
What is its value? Is $\sqrt{4} = 4^{\frac{1}{2}}$?

3. Read $\sqrt{m}$.
What does it mean?
Can you tell its value? Why not? (Because we do not know what m represents. As m may mean *anything*, $\sqrt{m}$ may mean *one* of the two equal factors of any number whatever.)

4. Read $\sqrt{5}$.
What does it mean?
Can you tell its value exactly?

---

* It is presumed that the pupil is familiar with this from arithmetic. He is expected to read "the square root of 4."

Is $\sqrt{5}$ more than 2?
Is it less than 3? Why?

5. Read $\sqrt[3]{8}$.
What does it mean?
What is its value?

6. Read $8^{\frac{1}{3}}$.
What does it mean?
Is $\sqrt[3]{8} = 8^{\frac{1}{3}}$?

7. Read $\sqrt[m]{x}$.
What does it mean?
Is $\sqrt[m]{x} = x^{\frac{1}{m}}$?

We will now give the full definition of an exponent, which should be very carefully studied, together with all that follows in this section. This is usually one of the most perplexing subjects, but we may hope by careful attention to strip it of its terrors.

140. An Exponent *is a small figure, letter or other symbol of number, written at the right and a little above another figure, letter or symbol of number.*

Point out the exponents in the expressions a^2, $b^{\frac{2}{3}}$, c^{-5}, $x^{-\frac{1}{2}}$, y^{-m}, d^{-n}, $e^{-\frac{m}{n}}$. What is the exponent of e in the last expression? Is it m? or n? or $\frac{m}{n}$? or $-\frac{m}{n}$? Be very careful and notice that the exponent may be a fraction, and may have a $-$ sign.

141. *There are* **Three kinds of Exponents,** *viz., positive integral, positive fractional, and negative.**

142. A Positive Integral Exponent *signifies that the number affected by it is to be taken as a factor as many times as there are units in the exponent. It is a kind of symbol of multiplication.*

HOW TO READ.—We have already learned about *Positive Integral Exponents* (8), and have now only to refresh our memory. 2^2 is read "2, second power," or "2 square;" so a^2 is read "a, second power," or "a square;" 5^3 is read "5, third power," or "5 cube;" so x^3 is read "x, third power," or "x cube;" m^4 is read "m, fourth power," etc.

If m *is an integer* x^m may be read "x, mth power;" so *if* n *is an integer*, y^n may be read "y, nth power." *But, if we do not know that the letter used represents a positive integer, it is absurd to read it so.* You will see how this is in (143).

If m represents a positive integer, x^m means xxx . . . etc., to m factors, just as x^5 means xx, etc., to 5 factors, *i. e.*, $xxxxx$. Of course we cannot write out *all* the factors of x^m, for we do not know how many m indicates.

1. How is y^5 read?
What does it mean?
Write it in another form ($yyyyy$).

2. What does x^4 mean?
Write it in another form. Read it.

3. Read $(a-b)^3$.

* It is not necessary to specify positive integral, negative integral, positive fractional, and negative fractional, as will appear from the treatment.

What does it mean?

Write it in another form: $(a-b)(a-b)(a-b)$.

4. How many times is $x+y$ a factor in $(x+y)^4$?

Write it in another form.

5. If m is an integer, what does a^m mean?

How is it read?

Write it in another form.

6. If n is an integer, what does $(a+x)^n$ mean?

How many times is $a+x$ a factor in $(a+x)^n$?

---

148. A Positive Fractional Exponent *indicates a power of a root, or a root of a power. The denominator specifies the root, and the numerator the power of the number to which the exponent is attached.*

HOW TO READ.—Such an expression as $8^{\frac{2}{3}}$ is read "8, exponent $\frac{2}{3}$." So $x^{\frac{2}{3}}$ is read "x, exponent $\frac{2}{3}$." This means x *with an* exponent $\frac{2}{3}$, the words "with an" being left out for brevity. $y^{\frac{m}{n}}$ is read "y, exponent $\frac{m}{n}$." Thus any form of exponent can be read.*

MEANING OF FRACTIONAL EXPONENTS.—According to the definition $8^{\frac{2}{3}}$ means the *second power*, of the *third root* of 8. Now the third root of 8 is 2, and the second power of 2 is 4. Hence $8^{\frac{2}{3}}$ is the same as 4, *i. e.*, $8^{\frac{2}{3}}=4$.

---

* The teacher needs to understand and fully explain, if there should be occasion, that such a reading as "8, two-thirds power," is absurd. There is no such thing as a two-thirds power. The definition of a *power* excludes it. Moreover, such a reading leads the pupil astray.

Again, $x^{\frac{m}{n}}$ means that one of the n equal factors of x is to be taken m times as a factor, just as $(16)^{\frac{3}{4}}$ signifies that one of the 4 equal factors of 16 (*i. e.*, 2) is to be taken 3 times as a factor (*i. e.*, $2 \cdot 2 \cdot 2$, or 8). Now as we do not know what number is represented by x, or what by m, or n, we cannot say any thing more definitely about $x^{\frac{m}{n}}$ than has been said; but we can tell that $(16)^{\frac{3}{4}}=8$.

1. Read $4^{\frac{3}{2}}$.
What does it mean?
Into how many factors is 4 to be resolved?
How many are to be taken?
What is the value of $4^{\frac{3}{2}}$?
What is the value of $(\sqrt{4})^3$?
Of $\sqrt{4^3}$? Are all alike?

2. Read $(125)^{\frac{2}{3}}$. Read $\sqrt[3]{(125)^2}$. Read $(\sqrt[3]{125})^2$. Is there any difference in value?

3. Read $7^{\frac{2}{3}}$.
What does it mean?
Can you tell its value exactly? Why not?
Can you find a number which, taken 3 times as a factor, makes just 7?
Is $7^{\frac{2}{3}}$ more, or less, than 4?
How do you know?
Is it more, or less, than 1? Why?

4. Read $y^{\frac{a}{b}}$. What does it mean? Write it in two

other forms. (See Exs. 1 and 2.) Can you tell the exact value of $y^{\frac{a}{b}}$.

5. Read $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $x^{\frac{1}{4}}$.
Write these in other forms.
Tell what they mean.

[NOTE.—A fraction used as an exponent *has not the same significance as a common fraction.* Thus $\frac{2}{3}$, as a common fraction, indicates the *sum* of 2 of the 3 equal *parts* of a quantity; but $\frac{2}{3}$ used as an exponent indicates the *product* of 2 of the 3 equal *factors* of a quantity. In all explanations this is to be kept clearly in view.]

---

144. A Negative Exponent, *i. e., one with the - sign before it, either integral or fractional, signifies the reciprocal of what the expression would be if the exponent were positive, i. e., had the + sign, or no sign at all before it.*

HOW TO READ.—The expression $a^{-\frac{2}{3}}$ is read "a, exponent $-\frac{2}{3}$." x^{-m} is read "x, exponent $-m$," etc.

MEANING OF NEGATIVE EXPONENTS.—According to the definition, 2^{-3} is the reciprocal of 2^3, *i. e.*, $2^{-3}=\frac{1}{2^3}$, or $\frac{1}{8}$. So $x^{-m}=\frac{1}{x^m}$; $y^{-\frac{1}{2}}=\frac{1}{y^{\frac{1}{2}}}$, or $\frac{1}{\sqrt{y}}$; $a^{-\frac{2}{3}}=\frac{1}{a^{\frac{2}{3}}}$, etc.

1. What is the value of $8^{-\frac{2}{3}}$
Of $(125)^{-\frac{4}{3}}$? Of $4^{-\frac{1}{2}}$? Of 3^{-2}?
Of $(32)^{-\frac{3}{5}}$? Of $(16)^{-\frac{3}{2}}$? Of 2^{-3}?
Of $8^{-\frac{1}{3}}$? Of $(27)^{-\frac{1}{3}}$? Of 1^{-1}?

2. What does x^{-n} mean? How read?

3. What does $5^{-\frac{m}{n}}$ mean? How read?

How Negative Exponents arise from Division.

1. What is $a^3 \div a$? $a^5 \div a^2$?

2. How do you divide a quantity affected by an exponent by the same quantity affected with another exponent?

3. What is $x^3 \div x^5$?

SOLUTION.—By the rule for division $x^3 \div x^5 = x^{3-5} = x^{-2}$. But $x^3 \div x^5$ may be written $\frac{x^3}{x^5}$ which is $\frac{1}{x^2}$. Hence we see that $x^{-2} = \frac{1}{x^2}$.

4. Solve as above $a^5 \div a^8$, and thus show that $a^{-3} = \frac{1}{a^3}$.

5. Solve $y \div y^5$ so as to show that $y^{-4} = \frac{1}{y^4}$.

6. What is $a^3 \div a^3$?

SOLUTION.—By the rule for division $a^3 \div a^3 = a^{3-3} = a^0$. But $a^3 \div a^3$ may be written $\frac{a^3}{a^3}$ which equals 1. Hence we see that $a^0 = 1$.

145. Any Quantity *with an* **Exponent** *0 is 1.*

7. Show as above, from $x^5 \div x^5$, that $x^0 = 1$.
Also from $b^m \div b^m$, that $b^0 = 1$.
Also from $y \div y$, that $y^0 = 1$.

How Fractional Exponents arise from Factoring.

1. What is one of the two equal factors of a^2? of a^4? of a^6?

146. *Thus we see that we can express one of the two equal factors of any number by dividing its exponent by* 2.

2. What is one of the two equal factors of a?

SOLUTION.—Since a is a^1, and since we can express one of its 2 equal factors by dividing its exponent by 2, we have $a^{\frac{1}{2}}$ as one of the two equal factors of a, *i. e.*, $\sqrt{a}=a^{\frac{1}{2}}$.

3. Show and explain as above, that $x^{\frac{1}{3}}$ and $\sqrt[3]{x}$ are the same.

4. Show and explain that $\sqrt{a^3}=a^{\frac{3}{2}}$.

SOLUTION.—By the definition of a square root we know that $\sqrt{a^3}$ means one of the two equal factors of a^3. But by the rules for division we know that a quantity can be resolved into two equal factors by dividing its exponent by 2. Hence one of the two equal factors of a^3 is $a^{\frac{3}{2}}$, *i. e.*, $\sqrt{a^3}=a^{\frac{3}{2}}$.

5. Show and explain as above that $\sqrt[3]{x^2}=x^{\frac{2}{3}}$.

SECTION XXV.

SIMILAR RADICALS.

147. A Radical Number *is an indicated root of a number. If the root can be extracted exactly, the quan-*

tity becomes **Rational**; *if the root cannot be extracted exactly, the expression is called* **Irrational, or Surd.**

Is $\sqrt{4}$ a radical? Is it rational, or irrational?

Is $\sqrt[3]{29a^6}$ a radical? Is it rational, or surd?

Is $\sqrt{5}$ a radical? Is it rational, or irrational?

Is $\sqrt{13a}$ a radical? Is it rational, or surd?

Is $\sqrt{a^2-b}$ rational, or irrational?

Is $\sqrt{a^2+2ab+b^2}$ rational, or irrational?

148. Similar Radicals *are like roots of like quantities.*

Thus $3\sqrt{2a}$, $5m\sqrt{2a}$, and $(a-b)\sqrt{2a}$, are all similar radicals, for it is the square root of $2a$ which is involved in each. But $3\sqrt{3a}$, and $4m\sqrt{5a}$ are not similar, nor are $5\sqrt{a}$ and $5\sqrt[3]{a}$. *In order to be similar the radical factor must be exactly the same in each.*

---

149. *It is frequently the case that dissimilar radicals can be reduced to similar ones. This is done upon the following*

PRINCIPLES.

1st. *The product of the same root of two or more quantities, equals the like root of their product.*

2d. *The quotient of the same root of two quantities equals the like root of their quotient.*

ILLUSTRATION.—The first principal asserts that $\sqrt{4} \times \sqrt{9} = \sqrt{36}$, as $\sqrt{4} \times \sqrt{9}$ is the product of the square roots of 4 and 9, and $\sqrt{36}$ is the square root of the product of these numbers. But

$\sqrt{4} \times \sqrt{9} = 2\times3 = 6$, and $\sqrt{36} = 6$. Therefore $\sqrt{4}\times\sqrt{9} = \sqrt{36}$. Is the *sum* of the square roots of two numbers equal to the square root of their *sum?* *i. e.*, is $\sqrt{4}+\sqrt{9} = \sqrt{4+9}$?

The second principle asserts that $\dfrac{\sqrt{9}}{\sqrt{4}} = \sqrt{\dfrac{9}{4}}$. That this is true we have learned in arithmetic, since we there learn that to extract the square root of a fraction we had but to extract the square root of the numerator and denominator separately. Is $\sqrt{49}-\sqrt{25} = \sqrt{49-25}$?

A study of the following examples will make these principles clear.

1. Extract the square root of 36, by resolving it into its prime factors.*

Solution. $36 = 4\times9 = 2\cdot2\times3\cdot3 = 2\cdot3\times2\cdot3 = 6\times6$. Hence 6 is one of the two equal factors of 36, and is therefore its square root.

2. Extract the square root of 225, by factoring, first into two factors which are square numbers.

3. Extract the cube root of 216, by factoring.

Solution. $216 = 27\times8 = 3\cdot3\cdot3\times2\cdot2\cdot2 = 2\cdot3\times2\cdot3\times2\cdot3 = 6\times6\times6$. Hence 6 is the cube root of 216 as it is one of the *three* equal factors of 216.

4. Extract the cube root of 3375, by observing that $3375 = 27\times125$.

150. *From these examples we see that a root may be extracted by extracting the root of the factors of the quantity and taking the product of these roots.*

* This process should have been taught in arithmetic. If it has not it should be made familiar here.

This is the same as Principle 1. But we will give a formal demonstration of this principle, as it is a very important one.

DEM.—That is $\sqrt[m]{x} \times \sqrt[m]{y} = \sqrt[m]{xy}$. This is evident from the fact that $\sqrt[m]{xy}$ signifies that xy is to be resolved into m equal factors. If now each factor, as x and y, be separately resolved into m equal factors and then the product of one factor from each be taken, there will be m such equal factors in xy. Thus $\sqrt[m]{x}$ is one of the m equal factors of x, and $\sqrt[m]{y}$ is one of the m equal factors of y. Hence $[\sqrt[m]{x} \times \sqrt[m]{y}] \times [\sqrt[m]{x} \times \sqrt[m]{y}] \times [\sqrt[m]{x} \times \sqrt[m]{y}]$ etc., to m factors of $\sqrt[m]{x} \times \sqrt[m]{y}$, makes up xy. Therefore $\sqrt[m]{x} \times \sqrt[m]{y} = \sqrt[m]{xy}$.

5. Extract the square root of $\frac{25}{16}$.
How do you do it?

6. Extract the cube root of $\frac{125}{27}$.
Is $\frac{\sqrt[3]{125}}{\sqrt[3]{27}} = \sqrt[3]{\frac{125}{27}}$?

A formal demonstration of the 2d Principle is as follows:

DEM.—Let m be an integer and x and y any numbers; we are to prove that $\sqrt[m]{x} \div \sqrt[m]{y}$, or $\frac{\sqrt[m]{x}}{\sqrt[m]{y}} = \sqrt[m]{\frac{x}{y}}$. Now, that $\frac{\sqrt[m]{x}}{\sqrt[m]{y}} = \sqrt[m]{\frac{x}{y}}$ is evident, since $\frac{\sqrt[m]{x}}{\sqrt[m]{y}}$ raised to the mth power, that is

$$\frac{\sqrt[m]{x} \times \sqrt[m]{x} \times \sqrt[m]{x} \times \sqrt[m]{x} \cdots \text{to } m \text{ factors}}{\sqrt[m]{y} \times \sqrt[m]{y} \times \sqrt[m]{y} \times \sqrt[m]{y} \cdots \text{to } m \text{ factors}} = \frac{x}{y};$$

whence it appears that $\frac{\sqrt[m]{x}}{\sqrt[m]{y}}$ is the mth root of $\frac{x}{y}$; or equals $\sqrt[m]{\frac{x}{y}}$.

151. The Degree *of a radical is determined by the number of factors into which the quantity is conceived to be resolved.*

Thus $\sqrt{a}$ is of the 2d degree, so also is $a^{\frac{1}{2}}$; $\sqrt[3]{x}$, or $x^{\frac{1}{3}}$, is of the 3d degree, etc.

To Reduce Dissimilar Radicals of the same Degree to Similar Radicals, when it can be done.

1. Reduce $\sqrt{128}$ and $\sqrt{72}$ to similar radicals.

SOLUTION.—Since $\sqrt{128}=\sqrt{4\times32}$, or $\sqrt{16\times8}$, or $\sqrt{64\times2}$, we can put it equal to $\sqrt{4}\,\sqrt{32}$, $\sqrt{16}\,\sqrt{8}$, or $\sqrt{64}\,\sqrt{2}$, and these are respectively $2\sqrt{32}$, $4\sqrt{8}$, and $8\sqrt{2}$. In like manner $\sqrt{72}=\sqrt{9\times8}$, or $\sqrt{36\times2}$, which are $\sqrt{9}\,\sqrt{8}$, and $\sqrt{36}\,\sqrt{2}$, or $3\sqrt{8}$ and $6\sqrt{2}$ respectively. Now we see that if we put $\sqrt{128}=\sqrt{64\times2}=8\sqrt{2}$, and $\sqrt{72}=\sqrt{36\times2}=6\sqrt{2}$, we have the radicals similar.

From this process we infer the following

152. RULE.—*Observe all the factors of the quantities under the radical signs which are perfect powers of the degree of the radicals. If then the quantities under the radicals can be factored so that each shall have a common factor, and a factor which is a perfect power of the same degree as the radical, extract the root of the latter factor, and remove it from under the sign in each case, leaving the common factor under each radical sign. The radicals will then be similar.*

2. Reduce $\sqrt{18}$ and $\sqrt{8}$ to similar radicals.

3. Reduce $\sqrt{27a^4x}$ and $\sqrt{3a^4x}$ to similar radicals.

Suggestion. $27a^4x=9a^4\times3x$, and $3a^4x=a^4\times3x$.

4. Reduce $\sqrt{108ax^2}$ and $\sqrt{48ax^2}$ to similar radicals.

5. Reduce $\sqrt{12}$, $2\sqrt{27}$, and $3\sqrt{75}$ to similar radicals.

Suggestion. $2\sqrt{27}=2\sqrt{9\times3}=2\sqrt{9}\sqrt{3}=2\cdot3\sqrt{3}=6\sqrt{3}$.

We observe that if the radical has a co-efficient, the root of the factor removed from under the radical must be multiplied into this co-efficient.

6. Reduce $5\sqrt{48a^3x^3}$ and $3\sqrt{108a^5x}$ to similar radicals.

7. Reduce $2\sqrt{175m^3x^2y}$ and $8\sqrt{252m^5y^3}$ to similar radicals.

SECTION XXVI.

ADDITION AND SUBTRACTION OF RADICALS.

1. Three times $\sqrt{2x}$ and 5 times $\sqrt{2x}$ make ho w many times $\sqrt{2x}$.

2. Seven times $\sqrt{5a}$ minus 3 times $\sqrt{5a}$ are how many times $\sqrt{5a}$?

3. $4\sqrt{7}+2\sqrt{7}-5\sqrt{7}$ are how many times $\sqrt{7}$?

4. $2\sqrt{12y}+3\sqrt{48y}$ are how much?

SOLUTION.—The radicals are dissimilar, and we cannot say that 2 times $\sqrt{12y}$ and 3 times $\sqrt{48y}$ make 5 times either, or anything. But let us see if the radicals cannot be made similar. $2\sqrt{12y}=4\sqrt{3y}$; and $3\sqrt{48y}=12\sqrt{3y}$. Now $4\sqrt{3y}+12\sqrt{3y}=16\sqrt{3y}$.

From these examples we readily infer the following

153. RULE.—*If the radicals are similar, add or subtract their co-efficients and to the result annex the common radical. If the radicals are dissimilar, reduce them to similar radicals, if possible, and proceed as before. If they cannot be made similar, they can only be connected with their proper signs like other dissimilar terms.*

DEM.—When the radicals are similar the radical factor is a common quantity and the co-efficients show how many times it is taken. Hence the sum, or difference, of the co-efficients, as the case may be, indicates how many times the common quantity is to be taken to produce the required result.

If the radicals are not similar, the reductions do not alter their values; hence the sum or difference of the reduced radicals, when they can be made similar, is the sum or difference of the radicals.

5. Add $3\sqrt{45}$ and $7\sqrt{20}$.

6. From $8\sqrt{125}$ take $2\sqrt{80}$.

7. Add $6\sqrt{27}$ and $9\sqrt{192}$.

8. From $9\sqrt{192}$ take $7\sqrt{75}$.

9. Add $2\sqrt{28a^3x}$ and $3\sqrt{252a^3x}$. *Sum*, $22a\sqrt{7ax}$.

10. From $5\sqrt{363a^3y^2}$ take $3\sqrt{243a^3y^2}$.

11. Add $a\sqrt{2b}$ and $c\sqrt{2b}$. *Sum*, $(a+c)\sqrt{2b}$.

12. From $2c\sqrt{xy}$ take $3b\sqrt{xy}$.

13. Add $10a\sqrt{28a^2x}$ and $5b\sqrt{63b^2x}$.

Sum, $5(4a^2+3b^2)\sqrt{7x}$.

14. From $3\sqrt{75m^3y^3}$ take $2\sqrt{27m^3y^3}$.

15. Add $\sqrt{13x^2}$ and $\sqrt{7y}$.

Suggestion.—As these radicals cannot be made similar they can be added only by connecting them with their proper signs.

16. From $\sqrt{10xy}$ take $2\sqrt{6x}$.

SECTION XXVII.

MULTIPLICATION AND DIVISION OF RADICALS.

Multiplication of radicals is effected by means of the 1st Principle of (**149**) and the following

PRINCIPLE.

154. *The numerator and denominator of a fractional exponent may be multiplied by the same number without affecting the value of the expression.*

ILLUSTRATION.—Thus $(64)^{\frac{2}{3}}=(64)^{\frac{4}{6}}$; for $(64)^{\frac{2}{3}}$ means the product of 2 of the three equal factors of 64, or 4•4. Now if each of the 3 equal factors of 64 is resolved into two equal factors, the whole will be resolved into six equal factors. But the 2 of the 3 equal factors indicated by $(64)^{\frac{2}{3}}$ will make 4 of the 6 equal factors produced when we resolve each of the 3 factors into 2 factors. Hence 2 of the 3 equal factors of 64 equals 4 of the 6 equal factors, or $(64)^{\frac{2}{3}}=(64)^{\frac{4}{6}}$.

1. Show that $(81)^{\frac{3}{2}}=(81)^{\frac{6}{4}}$.

2. Show that $(64)^{\frac{3}{2}}=(64)^{\frac{9}{6}}$.

Suggestion. $(64)^{\frac{3}{2}}=8\times8\times8$. Now resolving each of these factors into 3 equal factors, we have

$$(64)^{\frac{3}{2}}=2\cdot2\cdot2\times2\cdot2\cdot2\times2\cdot2\cdot2, \text{ or } (64)^{\frac{9}{6}},$$

since 3 of the 6 equal factors of any number make 1 of the two equal factors.

The following is a general demonstration of this important principle:

DEM.—In order to demonstrate this generally, we have to show that $x^{\frac{a}{b}}=x^{\frac{ma}{mb}}$. Now $x^{\frac{a}{b}}$ signifies the product of a of the b equal factors into which x is conceived to be resolved. If we now resolve each of these b equal factors into m equal factors, a of them will include ma of the mb equal factors into which x is conceived to be resolved. Hence ma of the mb equal factors of x equals a of the b equal factors.

[The student should notice the *analogy* between this explanation and that usually given in Arithmetic for reducing fractions to equivalent ones having a common denominator. It is not *identical*. See NOTE after Ex. 5, page 169.]

---

1. Multiply $a^{\frac{2}{3}}$ by $a^{\frac{3}{4}}$.

SOLUTION. $a^{\frac{2}{3}}=a^{\frac{8}{12}}$ since the product of 8 of the 12 equal fac

tors of a number is equal to the product of 2 of the 3 equal factors of the same number, for a number can be resolved into 12 factors by resolving it first into 3 and then each of these into 4.

In like manner $a^{\frac{3}{4}}=a^{\frac{9}{12}}$. [Give the explanation as above.]

We now have $a^{\frac{2}{3}}\times a^{\frac{3}{4}}=a^{\frac{8}{12}}\times a^{\frac{9}{12}}=\sqrt[12]{a^8}\times\sqrt[12]{a^9}$. But by the 1st Principle (149) $\sqrt[12]{a^8}\times\sqrt[12]{a^9}=\sqrt[12]{a^8\times a^9}=\sqrt[12]{a^{8+9}}=a^{\frac{8+9}{12}}=a^{\frac{17}{12}}$.

155. From this analysis we can infer the two following truths:

1*st. That a quantity affected with a fractional exponent may be multiplied by the same quantity affected with a fractional exponent by adding the exponents, the same as when the exponents are integral.* (See **27**.)

2*d. That radicals of different degrees can be multiplied by first reducing them to a common degree and then placing the common radical sign over the product of the quantities under it in both the factors.*

2. Multiply $a^{\frac{2}{3}}$ by $b^{\frac{3}{5}}$.

Suggestion.—**By (154) $a^{\frac{2}{3}}=a^{\frac{10}{15}}$, and $b^{\frac{3}{5}}=b^{\frac{9}{15}}$. Hence $a^{\frac{2}{3}}\times b^{\frac{3}{5}}=a^{\frac{10}{15}}\times b^{\frac{9}{15}}=\sqrt[15]{a^{10}}\times\sqrt[15]{b^9}=\sqrt[15]{a^{10}b^9}$. [The student should give a complete analysis of every step.]**

3. Multiply $5\sqrt{a}$ by $6\sqrt[3]{a}$.

Suggestion.—**The factors to be multiplied together are 5, $\sqrt{a}$, 6, and $\sqrt[3]{a}$. As the order of multiplication is immaterial (28), we may write $5\sqrt{a}\times 6\times\sqrt[3]{a}=5\cdot 6\sqrt{a}\times\sqrt[3]{a}=30\sqrt{a}\sqrt[3]{a}=30a^{\frac{1}{2}}a^{\frac{1}{3}}=30a^{\frac{5}{6}}$, the last operation being performed by 155 (1st).**

4. Multiply $3\sqrt{15}$ by $\sqrt{6}$.

Suggestion.—By (**149**, 1st) $3\sqrt{15}\times\sqrt{6}=3\sqrt{15\times 6}=3\sqrt{90}$. $3\sqrt{90}$ can be reduced by taking the factor 9 from under the radical sign. Thus $3\sqrt{90}=3\sqrt{9\times 10}=9\sqrt{10}$. This is the product in the simplest form.

156. A radical ***is said to be in its simplest form when the quantity under the radical sign is the smallest possible integer.***

5. Multiply $\sqrt{\frac{2}{3}}$ by $\sqrt{\frac{3}{5}}$.

Suggestion.—By (**149**, 1st) $\sqrt{\frac{2}{3}}\times\sqrt{\frac{3}{5}}=\sqrt{\frac{2}{3}\times\frac{3}{5}}=\sqrt{\frac{2}{5}}$. Now this is the product, but it is not in its *simplest form*, since there is a fraction under the radical sign. But we observe that $\sqrt{\frac{2}{5}}=\sqrt{\frac{2}{5}\times\frac{5}{5}}=\sqrt{\frac{10}{25}}=\sqrt{\frac{1}{25}\times 10}=\frac{1}{5}\sqrt{10}$, which is the product in its simplest form.

6. Multiply $\sqrt{\frac{2}{3}}$ by $\sqrt{\frac{3}{4}}$. *Prod.*, $\frac{1}{2}\sqrt{2}$.

7. Show that $\sqrt{\frac{1}{2}}=\frac{1}{2}\sqrt{2}$.

8. Multiply $3\sqrt{5}$ by $2\sqrt[3]{2}$. *Prod.*, $6\sqrt[6]{500}$.

9. Multiply $2\sqrt{3xy}$ by $5\sqrt{3xy^3}$. *Prod.*, $30xy^2$.

10. Multiply $\sqrt{\dfrac{2xy}{3a}}$ by $\sqrt{\dfrac{3bx}{2y}}$. *Prod.*, $\dfrac{x}{a}\sqrt{ab}$.

11. Multiply $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{1}{3}}$. *Prod.*, $\frac{1}{6}\sqrt{6}$.

12. Multiply $5\sqrt{5}$ by $3\sqrt{8}$.

13. Multiply $3\sqrt{2}$ by $2\sqrt{2}$.

14. Multiply $\sqrt{2}$ by $\sqrt{8}$.

15. Multiply $\sqrt[3]{2}$ by $\sqrt[3]{4}$.

16. Multiply $\sqrt{a}$ by $\sqrt{b}$.

17. Multiply $2\sqrt{ab}$ by $3\sqrt{ac}$.

18. Multiply $5\sqrt[3]{a^2c^2}$ by $a\sqrt{ac}$.

Queries.—In order to change $\sqrt[3]{\ }$ for $\sqrt[6]{\ }$ what must we do to the quantities under the sign? What in order to change $\sqrt{\ }$ for $\sqrt[6]{\ }$?

19. Multiply 3 by $\sqrt{\frac{2}{3}}$.

Suggestion.—We can put 3 under the square root sign by squaring it, since $\sqrt{9}=3$. Hence we have $3\times\sqrt{\frac{2}{3}}=\sqrt{9}\times\sqrt{\frac{2}{3}}=\sqrt{9\times\frac{2}{3}}=\sqrt{6}$.

20. Introduce the co-efficient under the radical sign in $5a\sqrt{\dfrac{b}{15a}}$ and put the result in the simple form.

Result, $\frac{1}{3}\sqrt{15ab}$.

21. Multiply $\sqrt{a}+\sqrt{b}$ by $\sqrt{a}+\sqrt{b}$.

PROCESS.

$$\begin{array}{l}
\sqrt{a}+\sqrt{b} \\
\underline{\sqrt{a}+\sqrt{b}} \\
a+\sqrt{ab} \\
\underline{\qquad\sqrt{ab}+b} \\
a+2\sqrt{ab}+b
\end{array}$$

This result should also be known as the square of the binomial $\sqrt{a}+\sqrt{b}$ by (40).

22. Multiply $\sqrt{a}-\sqrt{b}$ by $\sqrt{a}+\sqrt{b}$ as above. What inspection should give the product?

23. Multiply $2-3\sqrt{5}$ by $1+2\sqrt{5}$.

PROCESS.

$$\begin{array}{l} 2-3\sqrt{5} \\ 1+2\sqrt{5} \\ \hline 2-3\sqrt{5} \\ \quad 4\sqrt{5}-30 \\ \hline 2+\sqrt{5}-30, \text{ or } \sqrt{5}-28. \end{array}$$

24. Square $\sqrt{a^2-x^2}$.

Suggestion.—Since to square a quantity is to multiply it by itself, we have $\sqrt{a^2-x^2}\times\sqrt{a^2-x^2}=\sqrt{(a^2-x^2)(a^2-x^2)}=\sqrt{(a^2-x^2)^2}=a^2-x^2$. That $\sqrt{(a^2-x^2)^2}=a^2-x^2$ is evident from the fact that a square root is one of the two equal factors of a number.

157. *To square a quantity affected with the sign $\sqrt{\ }$ is simply to drop the sign.*

25. Square $\sqrt{2-x}$.

26. Multiply $\sqrt{a-b}$ by $\sqrt{a+b}$. *Prod.*, $\sqrt{a^2-b^2}$.

27. Multiply $\sqrt{1-x}$ by $\sqrt{x}$. *Prod.*, $\sqrt{x-x^2}$.

28. Square $2-\sqrt{3}$ both by actual multiplication and by observing that it is the square of a binomial.

29. Square $\sqrt{1-x}+\sqrt{1+x}$.

Suggestion.—This is the square of a binomial. Hence by (**40**), we have "the square of the 1st term," or $1-x$, "plus twice the product of the two terms," or $2\sqrt{1-x^2}$, "plus the square of the second term," or $1+x$. Hence

$$\left\{\sqrt{1-x}+\sqrt{1+x}\right\}^2=1-x+2\sqrt{1-x^2}+1+x=2+2\sqrt{1-x^2}.$$

30. Square as above $\sqrt{a+x^2}-\sqrt{a-x^2}$.

Square, $2a-2\sqrt{a^2-x^4}$.

Examples in Division.

158. No new principles are needed to enable us to effect division of radicals.

1. Divide $8\sqrt{ab}$ by $2\sqrt{a}$.

Solution.—We may write $\frac{8\sqrt{ab}}{2\sqrt{a}}=\frac{8}{2}\frac{\sqrt{ab}}{\sqrt{a}}=4\sqrt{\frac{ab}{a}}=4\sqrt{b}$. See (**149**, 2d).

2. Divide $8\sqrt{108}$ by $2\sqrt{6}$. *Quot.*, $12\sqrt{2}$.

3. Divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$.

Suggestion.—We have $\frac{\frac{1}{2}\sqrt{5}}{\frac{1}{3}\sqrt{2}}=\frac{3}{2}\times\sqrt{\frac{5}{2}}=\frac{3}{4}\sqrt{10}$.

4. Divide $4\sqrt[3]{ax}$ by $3\sqrt{xy}$.

Suggestion.—We have $\frac{4\sqrt[3]{ax}}{3\sqrt{xy}}=\frac{4}{3}\times\frac{\sqrt[6]{a^2x^2}}{\sqrt[6]{x^3y^3}}=\frac{4}{3}\sqrt[6]{\frac{a^2x^2}{x^3y^3}}=$ $\frac{4}{3}\sqrt[6]{\frac{a^2}{xy^3}}=\frac{4}{3xy}\sqrt[6]{a^2x^5y^3}$.

5. Divide $\sqrt{54}$ by $5\sqrt{2}$.

6. Divide $\sqrt{\frac{2}{3}}$ by $\sqrt[3]{\frac{5}{3}}$.

Suggestion. $\sqrt{\frac{2}{3}} \div \sqrt[3]{\frac{5}{3}} = \sqrt[6]{\frac{2^3}{3^3} \times \frac{3^2}{5^2}} = \sqrt[6]{\frac{2^3}{3.5^2}} = \sqrt[6]{\frac{2^3.3^5.5^4}{3^6.5^6}} = \frac{1}{15}\sqrt{2^3.3^5.5^4}$.

7. Divide $15ab\sqrt{6xy}$ by $5b\sqrt{2y}$.

8. Divide $6x\sqrt{48y^4}$ by $3\sqrt{4y^2}$.

9. Divide $\sqrt{\frac{2}{3}}$ by $\sqrt{\frac{1}{2}}$. *Quot.*, $\frac{2}{3}\sqrt{3}$.

10. Divide $12x^2y\sqrt{72x^2y^2}$ by $6x\sqrt{8xy}$.

11. Divide $\sqrt{20a^8}$ by $\sqrt{5a}$.

QUADRATIC EQUATIONS. PURE.

SECTION XXVIII.

159. The Degree *of an equation is determined by the highest number of unknown factors occurring in any term when the equation is freed from radicals or fractional exponents as affecting the unknown quantity.*

Thus $ax-bx^2=c+x^3$, is of the 3d degree; $a^2x-4x=12$ is of the 1st degree; $x^2y^2=18$ is of the 4th degree, etc.

160. A Simple Equation *is an equation of the first degree.*

Thus $y=ax+b$ is a simple equation, as also is $\frac{x-3}{2}+4x=\frac{1}{3}x+5$.

161. A Quadratic Equation *is an equation of the second degree.*

ILLUSTRATION. $x^2+3x=5$ is a quadratic, as is also $x^2=a$, or $xy=b$, or $x^2+y^2=7$.

162. Quadratic Equations *are distinguished as* **Pure** (*called also* **Incomplete**), *and* **Affected** (*called also* **Complete**).

163. A Pure Quadratic Equation *is an equation which contains no power of the unknown quantity but the second.*

Thus $ax^2+b=cd$, and $x^2-3b=102$, are pure quadratics.

164. A Root *of an equation is a quantity which, substituted for the unknown quantity, satisfies the equation. It is the value of the unknown quantity.*

1. What is the value of x in the equation $x^2=9$?

SOLUTION.—As the square roots of equal quantities are equal, we can extract the square root of each member and not destroy the equation. Hence we have $x=\pm 3$. It seems, then, that x has *two values*, viz., $x=+3$, and $x=-3$. This is evidently true since the square of $+3$ is 9, and the square of -3 is also 9.

165. A Pure Quadratic Equation *always has two roots, numerically the same, but with opposite signs.*

2. What are the roots of $x^2-36=\frac{x^2}{4}+12$?

Solution.—Clearing the equation of fractions, transposing and uniting terms, and dividing by 3, as in simple equations, we have

$$x^2=64.$$

Whence $x=+8$, and -8.

Solve the following:

3. $6x^2-48-2x^2=96$.

4. $2x^2+9=81$.

5. $x^2-3=\frac{4x^2+18}{9}$.

6. $(2x-5)^2=x^2-20x+73$.

7. $\frac{7x^2-25}{2}=2x^2-6\frac{1}{2}$.

8. $ax^2-b=1$. $\qquad x=\pm\sqrt{\frac{1+b}{a}}$.

9. $a^2x^2-b^2=0$. $\qquad x=\pm\frac{b}{a}$.

Problems.

1. There is a number such that by adding 5 to it for one factor, and subtracting 5 from it for another factor, we may obtain 96 for the product. What is that number?

Suggestion.—The equation is $(x+5)(x-5)=96$, or

$$x^2-25=96.$$

From this $x=+11$, and $x=-11$. Which of these shall we take? In truth, both will fulfill the conditions. But as we usually refer to positive quantities only in such questions, it is customary in elementary inquiries to neglect the negative roots. They will be neglected in these problems.

2. Find two numbers such that their product shall be 750, and the quotient of the greater divided by the less, $3\frac{1}{3}$.

3. Find a number such that if $\frac{1}{7}$ and $\frac{1}{8}$ of it be multiplied together, and the product divided by 3, the quotient will be $298\frac{2}{3}$. *Ans.*, 224.

4. Find two numbers which shall be to each other as 2 to 3, and the sum of whose squares shall be 208.
Ans., 8 and 12.

5. A merchant bought 2 pieces of cloth, which together measured 36 yards. Each piece cost as many quarter-dollars per yard as there were yards in it, and the entire cost of one piece was 4 times the cost of the other. How many yards were there in each piece?
Ans., 24 in one, and 12 in the other.

6. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy's coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in five lines. What was the number of men? *Ans.*, 4550.

Suggestion.—If we call the first line x men, there would have been $x+5$ lines, and hence x^2+5x men in the detachment. The equation is $x^2+5x=5x+4225$.

7. A man lent a certain sum of money at 6 per cent.

a year, and found that if he multiplied the principal by the number representing the interest for 8 months, the product would be \$900. Required the principal.

Principal, \$150.

Suggestion.—The interest for 1 year is $\frac{6x}{100}$, x being the principal; for $\frac{2}{3}$ of a year it is $\frac{2}{3}$ of $\frac{6x}{100}$, or $\frac{x}{25}$.

8. Find three uumbers in the ratio of 2, 3, and 5, the sum of whose squares is 342. *Numbers,* 6, 9, and 15.

Suggestion.—Let $2x$, $3x$, and $5x$ represent the numbers.

9. What two numbers are those which are to each other as 3 to 4, and the difference of whose squares is 28?

Numbers, 6 and 8.

10. Find three numbers in the ratio of m, n, and r, the sum of whose squares is s.

$$\textit{Numbers, } m\sqrt{\frac{s}{m^2+n^2+r^2}},\ n\sqrt{\frac{s}{m^2+n^2+r^2}},\ r\sqrt{\frac{s}{m^2+n^2+r^2}}.$$

11. What two numbers are those which are to each other as m to n, and the difference of whose squares is s?

$$\textit{Numbers, } m\sqrt{\frac{s}{m^2-n^2}},\ n\sqrt{\frac{s}{m^2-n^2}}.$$

12. The number of rods in the length and breadth of a rectangular field are in the ratio of 4 to 3, and the length of the diagonal is 100 rods. How many acres are in the field? *Ans.,* 30 acres.

13. Robert has three equal square lots. If he had 193 square rods more, he would have as much land as would be in one square lot whose sides are each 25 rods. What is the length of each side of the three equal square lots?

Ans., 12 rods.

14. There are two numbers whose sum is 17, and the less divided by the greater is to the greater divided by the less as 64 : 81. What are the numbers? *Ans.*, 8 and 9.

15. What are the two numbers whose product is a and quotient b?

Ans., $\sqrt{ab}$ and $\sqrt{\dfrac{a}{b}}$.

16. What two numbers are as $m : n$, the sum of whose squares is a?

Ans., $\dfrac{m\sqrt{a}}{\sqrt{(m^2+n^2)}}$ and $\dfrac{n\sqrt{a}}{\sqrt{(m^2+n^2)}}$.

QUADRATIC EQUATIONS AFFECTED.

SECTION XXIX.

166. An Affected Quadratic *equation is an equation which contains terms of the second degree and also of the first, with respect to the unknown quantity.*

Thus $x^2-2x=8$, $x-\frac{1}{3}x^2=\dfrac{x^2+x}{5}$, and $x-1=\dfrac{x^2+3x}{7}$, are affected quadratics.

167. In order to solve an affected quadratic equation, we need to observe carefully the square of a binomial. To this we will attend before attempting the solution of such an equation.

1. What is the square of $x+2$?

Of how many terms does the square of a binomial consist?

What is the first term?

What the second term?

What the third term?

2. What is the square of $x-5$?

Same questions as above.

3. What is the square of $x+a$?

Same questions as above.

4. What is the square of $x-a$?

Same questions as above.

---

5. If x^2+4x is the first two terms of the square of a binomial, what is the third term?

What is the first term of the square of a binomial? Then what is the first term of the binomial, the first term of whose square is x^2?

What is the second term of the square of a binomial?

Then half the second term is what?

If then $2x$ is the product of the two terms of a binomial, and the first term of the binomial is x, what is the second term of the binomial?

6. If x^2-18x is the first two terms of the square of a binomial, what is the third term?

Will $x-4$ squared give x^2-18x for the first two terms of the square?

Will $x-2$ squared give these?

Will $x+3$?

Will $x-3$?

7. If x^2-6x is the first two terms of the square of a binomial, what is the third term?

8. What is the square of $x+\frac{1}{2}a$?

9. What is the square of $x-\frac{1}{2}a$?

10. If x^2+ax is the first two terms of the square of a binomial, what is the third term?

What is the binomial of which x^2+ax is the first two terms of its square?

11. What is the binomial of which x^2-ax is the first two terms of its square?

12. Is the first term of the square of a binomial ever $-$?

Is the third term of the square of a binomial ever $-$?

When is the middle term of the square of a binomial $+$, and when $-$?

168. *If x^2+ax, or x^2-ax, is the first two terms of the square of a binomial, the third term of this square is* **the square of half the co-efficient of** x *and is always* $+$.

13. According to this principle tell what the *Completed Square* is in each of the following:

x^2+8x. *Completed Square,* $x^2+8x+16$.

x^2-10x. *Completed Square,* $x^2-10x+25$.

x^2+12x.

x^2-12x.

x^2+20x.

x^2+16x.

x^2-5x. *Completed Square,* $x^2-5x+\frac{25}{4}$.

x^2+3x.

x^2+x. What is half the co-efficient of x?

x^2-7x.

x^2-x.

x^2-3mx. *Completed Square,* $x^2-3mx+\frac{9m^2}{4}$.

$x^2+\frac{2a}{b}x$. *Completed Square,* $x^2+\frac{2a}{b}x+\frac{a^2}{b^2}$.

$x^2-\frac{m}{2n}x$. *Completed Square,* $x^2-\frac{m}{2n}x+\frac{m^2}{16n^2}$.

$x^2+(a-b)x$. What is the square of $\frac{1}{2}$ the co-efficient of x?

$x^2-\frac{2m+4}{a}x$.

$x^2+\frac{a+1}{2m}x$. *Completed Square,* $x^2+\frac{a+1}{2m}x+\frac{(a+1)^2}{16m^2}$.

14. What is the square root of the completed square of the following:

x^2-6x? *Square Root of C. Sqr.,* $x-3$.

x^2+24x? *Square Root of C. Sqr.,* $x+12$.

x^2+14x?

x^2-2x?

x^2-x?

x^2+x?

x^2-11x? *Square Root of C. Sqr.,* $x-\frac{11}{2}$.

x^2+9x?

x^2-ax?

$x^2+\frac{2}{3}x$?

$x^2-\frac{a}{b}x$?

$x^2+\frac{2m-2}{a}x$?

$x^2-\frac{a^2-1}{b}x$? *Square Root of C. Sqr.*, $x-\frac{a^2-1}{2b}$.

$x^2+\frac{2m}{a}x$?

$x^2-7\frac{1}{2}x$?

$x^2-\frac{3}{5}x$?

y^2-amy?

y^2+m^2y?

$y^2-\frac{1}{2}n^2y$?

169. *If* x^2+ax, *or* x^2-ax, *is the first two terms of the square of a binomial, the binomial is* x +, *or* −, *half the co-efficient of* x *in the given expression,* + *when the sign of* **ax** *is* +, *and* − *when the sign is* −.

Solution of Affected Quadratics.

1. Given $x^2+10x=24$ to find the value of x.

Solution.—Adding 25 to each member, which will not destroy the equation, it becomes

$$x^2+10x+25=49.$$

Extracting the square root of each member, since the square roots of equal quantities are equal, we have

$$x+5=\pm 7.$$

Hence, $x=7-5$, and $-7-5$,

or $x=2$, and -12.

170. We see ***from this solution that an affected quadratic has two roots, or values of the unknown quantity.***

2. Find the values of x in the equation $x^2-6x=135$.

SOLUTION.—Completing the square of the first member by adding 9 to it, and also adding 9 to the second member so as to preserve the equality of the members, we have

$$x^2-6x+9=144.$$

Extracting the square root of each member, which does not destroy the equation, since the square roots of equal quantities are equal, we have

$$x-3=\pm 12.$$

Whence, $\quad x=12+3$, and $-12+3$,

or $\quad x=15$, and -9.

3. Solve $\dfrac{x^2+4}{2x}=\dfrac{x+6}{4}$.

SOLUTION.—Clearing the equation of fractions, we have

$$2x^2+8=x^2+6x.$$

Transposing the terms containing x into the first member and the known terms into the second, we have

$$2x^2-x^2-6x=-8.$$

Uniting similar terms, we obtain

$$x^2-6x=-8.$$

Solving this as before, we find $x=4$, and 2.

4. Solve $9x+\dfrac{1}{x}=\dfrac{29}{x}+4$.

SOLUTION.—Clearing of fractions, transposing, and uniting terms, we have

$$9x^2-4x=28.$$

Dividing each member by 9, we obtain

$$x^2-\tfrac{4}{9}x=\tfrac{28}{9}.$$

Completing the square,

$$x^2-\frac{4}{9}x+\left(\frac{2}{9}\right)^2=\frac{28}{9}+\frac{4}{81}=\frac{256}{81}.$$

Extracting the square root of each member, we have

$$x-\frac{2}{9}=\pm\frac{16}{9}.$$

Whence, $x=2$, and $-\frac{14}{9}$.

Note.—It is not best actually to square the half co-efficient of x as added to the first member, but only to *indicate* its square.

From these examples we infer the following rule for solving affected quadratics:

171. RULE.—1st. *Reduce the equation to the form* $x^2+ax=b$.

2d. *Add the square of half the co-efficient of the second term to both members of the equation.*

3d. *Extract the square root of each member, thus producing a simple equation from which the value of the unknown quantity is found by simple transposition.*

Dem.—By definition, an Affected Quadratic Equation contains but three kinds of terms, viz., terms containing the square of the unknown quantity, terms containing the first power of the unknown quantity, and *known* terms. Hence each of the three kinds of terms may, by clearing of fractions, transposition, and uniting, as the particular example may require, be united into one, and the results arranged in the order given. If, then, the first term, *i. e.*, the one containing the square of the unknown quantity, has a co-efficient other than unity, or is negative, its co-efficient can be rendered unity or positive without destroying the equation, by dividing both members by whatever co-efficient this term may chance to have after the first reductions. The equation will then take the form $x^2\pm ax=\pm b$. Now adding $\left(\frac{a}{2}\right)^2$ to the first member, makes it a perfect square $\left(\text{the square of } x\pm\frac{a}{2}\right)$, since a tri-

nomial is a perfect square when one of its terms (the middle one, ax, in this case) is $\pm$ twice the product of the square roots of the other two, these two being both positive. But, if we add the square of half the co-efficient of the second term to the first member to make it a complete square, we must add it to the second member to preserve the equality of the members. Having extracted the square root of each member, these roots are equal, since like roots of equals are equal. Now, since the first term of the trinomial square is x^2, and the last $\left(\frac{a^2}{4}\right)$ does not contain x, its square root is a binomial consisting of $x \pm$ the square root of its third term, or half the co-efficient of the middle term, and hence a known quantity. The square root of the second member can be taken exactly, approximately, or indicated, as the case may be. Finally, as the first term of this resulting equation is simply the unknown quantity, its value is found by transposing the second term.

---

172. Solve the following affected quadratics, verifying those to which the answers are not given.

1. $x^2-15=45-4x$. $\qquad x=6$, and -10.

2. $x^2-6x+9=1$.

3. $x^2=8x+9$.

4. $5x-23=\frac{25-3x}{x}$. $\qquad x=5$, and -1.

VERIFICATION.—To verify the value $x=5$, we have

$$5 \cdot 5-23=\frac{25-3 \cdot 5}{5}.$$

Now $5 \cdot 5-23$ is 2; and $\frac{25-3 \cdot 5}{5}$ is also 2, hence the equation is true for $x=5$.

To verify the value $x=-1$, we have

$$5\times(-1)-23=\frac{25-3\times(-1)}{-1}.$$

Now $5\times(-1)-23$ is $-5-23$, or -28.

And $\frac{25-3\times(-1)}{-1}$ is $\frac{25+3}{-1}$, or -28.

Hence the equation is true for $x=-1$.

5. $x+4=13-\frac{7x-8}{x}$. $x=4$, and -2.

6. $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$. $x=7$, and $-6\frac{1}{3}$.

7. $\frac{22-x}{20}=\frac{15-x}{x-6}$.

8. $\frac{x}{x+8}=\frac{x+3}{2x+1}$.

9. $\frac{x}{x+60}=\frac{7}{3x-5}$.

10. $3x^2-408=2x$.

11. $-12+61x=5x^2$.

12. $2x+\frac{3x-6}{2}=5x-\frac{3x-3}{x-3}$. $x=4$, and -1.

13. $3x-\frac{1121-4x}{x}=2$.

14. $x^2+6x=55$.

15. $x^2-5x=7$.

Solution.—Completing the square, we have

$$x^2-5x+\left(\frac{5}{2}\right)^2=\frac{53}{4}.$$

Extracting the square root, $x-\frac{5}{2}=\pm\sqrt{\frac{53}{4}}=\pm\frac{1}{2}\sqrt{53}.$

Hence $x=\frac{5}{2}\pm\frac{1}{2}\sqrt{53}.$

In such a case as this, i. e., *when the second member of the equation is not a perfect power, after the square of the first member has been completed, we may leave the result in the form above, or extract the square root of the member under the radical to any required degree of accuracy.*

Thus, $\sqrt{53}=7.28+$. Whence $x=\frac{5+7.28}{2}$, and $\frac{5-7.28}{2}$, or 6.14, and -2.14, approximately.

16. $x^2+4x=11.$ $x=1.8+$, and $-5.8+$.

17. $\frac{x-2}{5}=\frac{2}{x}+1.$ $x=8.22$, and -1.22, nearly.

18. $3-5x+2x^2=7.$ $x=\frac{5\pm\sqrt{57}}{4}.$

19. $\frac{2x^2-5}{3}=\frac{x^2+7x}{4}.$

20. $x^2-4x=-20.$ $x=2\pm\sqrt{-16}.$

NOTE.—Such an expression as $\sqrt{-16}$, *i. e.*, the indicated square root of a negative quantity, is called *An Imaginary Quantity.* You observe that we cannot obtain the square root of a negative quantity, for the square root of a quantity is a quantity which multiplied by itself produces the given quantity. Now, no real quantity multiplied by itself produces a *negative* quantity. Thus, if we attempt to get the square root of -16, what shall we call it? Is it $+4$? No; since $+4$, squared, is $+16$. Is it, then, -4? No; for -4, squared, is $+16$. But, $(+4)\times(-4)=-16$. Is not one of these factors, then, the square root of -16? No; since the factors must be equal, and $+4$ and -4 are not equal.

178. An imaginary quantity *is an indicated even root of a negative quantity.*

21. $x^2-10x=-40$. $\qquad x=5\pm\sqrt{-15}$.

22. $x^2+\ 8x=-50$. $\qquad x=-4\pm\sqrt{-34}$.

23. $\dfrac{x}{x+a}=\dfrac{b}{x-b}$.

SOLUTION.—Clearing of fractions, $\qquad x^2-bx=bx+ab$.
Transposing and uniting, $\qquad x^2-2bx=ab$.
Completing the square, $\qquad x^2-bx+b^2=ab+b^2$.
Extracting square root, $\qquad x-b=\pm\sqrt{ab+b^2}$.
Transposing, $\qquad x=b\pm\sqrt{ab+b^2}$.

24. $a^2+b^2-2bx+x^2=\dfrac{m^2x^2}{n^2}$.

SOLUTION.—Clearing of fractions, $a^2n^2+b^2n^2-2bn^2x+n^2x^2=m^2x^2$
Transposing and uniting, $\qquad (n^2-m^2)x^2-2bn^2x=-a^2n^2-b^2n^2$
Dividing by the co-efficient of x^2, $\qquad x^2-\dfrac{2bn^2}{n^2-m^2}x=-\dfrac{a^2n^2+b^2n^2}{n^2-m^2}$.

Completing the square,

$$x^2-\frac{2bn^2}{n^2-m^2}x+\left(\frac{bn^2}{n^2-m^2}\right)^2=\frac{b^2n^4}{(n^2-m^2)^2}-\frac{a^2n^2+b^2n^2}{n^2-m^2}.$$

Uniting terms in second member,

$$x^2-\frac{2bn^2}{n^2-m^2}x+\left(\frac{bn^2}{n^2-m^2}\right)^2=\frac{n^2(a^2m^2+b^2n^2-a^2n^2)}{(n^2-m^2)^2}.$$

Extracting square root of each member,

$$x-\frac{bn^2}{n^2-m^2}=\pm\frac{n}{n^2-m^2}\sqrt{a^2m^2+b^2n^2-a^2n^2}.$$

Transposing and factoring,

$$x=\frac{n}{n^2-m^2}\left\{bn\pm\sqrt{a^2m^2+b^2n^2-a^2n^2}\right\}.$$

25. $ax^2+bx=c.$ $\qquad x=\dfrac{-b\pm\sqrt{4ac+b^2}}{2a}.$

26. $2ax-x^2=-2ab-b^2.$ $\qquad x=2a+b$, and $-b$.

27. $x^2+ax=b.$

28. $x^2-ax=b.$

29. $x^2+ax=-b.$

30. $x^2-ax=-b.$

SECTION XXX.

PROBLEMS PRODUCING AFFECTED QUADRATICS.

1. Divide the number 56 into two such parts that their product shall be 640.

Suggestion.—The equation is $(56-x)x=640$. The parts are 40 and 16.

2. Divide a into two such parts that their product shall be m. Deduce from the result the results in the preceding.

Suggestion.—The equation is $(a-x)x=m$. The parts are $\dfrac{a+\sqrt{a^2-4m}}{2}$ and $\dfrac{a-\sqrt{a^2-4m}}{2}$.

3. The difference between two numbers is 6, and the sum of their squares is 50. What are the numbers?

Ans., 7 and 1.

4. The difference between two numbers is d, and the

sum of their squares is s. What are the numbers? Deduce the results in the last from these answers.

$$Ans., \frac{-d+\sqrt{2s-d^2}}{2}, \text{ and } \frac{d+\sqrt{2s-d^2}}{2}.$$

5. A nursery-man planted 8400 trees at equal distances, in the form of a rectangle, having 50 trees more in front than in depth. What was the number in front?

Ans., 120 trees.

6. Divide s into two such parts that their product shall be n times their difference. Find from the literal or general answers, the answers when 30 is the number and their product is 8 times their difference.

Literal Results, $x=\frac{1}{2}\left(s+2n\pm\sqrt{s^2+4n^2}\right)$.

Queries.—Would it be consistent with the problem to take the value of x as $\frac{1}{2}\left(s+2n+\sqrt{s^2+4n^2}\right)$? Would this make x, which is one of the parts of s, greater or less than s? x being one of the parts, and its value being $\frac{1}{2}\left(s+2n-\sqrt{s^2+4n^2}\right)$, the other part, $s-x$, is $\frac{1}{2}\left(s-2n+\sqrt{s^2+4n^2}\right)$.

7. The ages of a man and his wife amount to 42 years, and the product of their ages is 432. What is the age of each? *Ans.*, Man's, 24 years; wife's, 18 years.

Queries.—Which of the above problems is essentially the same as this? Which of the general (literal) answers will give the answer to this by merely substituting the numerical values as here given?

8. A is 4 years older than B; and the sum of the squares of their ages is 976. What are their ages?

Ans., A's age, 24 years; B's, 20 years.

Queries.—Which of the above problems is essentially the same as this? Which of the general solutions above covers this?

9. A merchant has a piece of broadcloth and a piece of silk. The number of yards in both is 110; and if the square of the number of yards of silk be subtracted from 80 times the number of yards of broadcloth, the difference will be 400. How many yards are there in each piece? *Ans.*, 60 of silk; 50 of broadcloth.

10. A merchant bought a piece of cloth for $45, and sold it for 15 cents more per yard than he paid. Though he gave away 5 yards, he gained $4.50 on the piece. How many yards did he buy, and at what price per yard? *Ans.*, 60 yards, at 75 cents per yard.

Suggestion.—Let $x =$ number of yards; the equation is $\left(\frac{4500}{x}+15\right)(x-5)=4500+450$.

11. A person being asked his age, answered, "My mother was 20 years old when I was born, and her age multiplied by mine exceeds our united ages by 2500." What was his age? *Ans.*, 42 years.

12. The length of a certain field exceeds its width by 8 rods; and its area is 768 rods. What are the dimensions of the field? *Ans.*, Length, 32 rods; width, 24 rods.

Queries.—Which of the above problems is essentially the same as this? Deduce the answers to this from the general results in the corresponding problem above.

13. A man bought a number of sheep for $240, and sold them again for $6.75 apiece, gaining by the bargain as much as 5 sheep cost him. How many sheep did he buy? *Ans.*, 40.

Suggestion.—Letting x=number of sheep, $\frac{24000}{x}$ cents=cost of 1 sheep, and $\frac{120000}{x}$= amount gained by the bargain.

14. A person invested a certain sum of money for goods, which he sold again for \$24, and thereby lost as many per cent. as equaled the number of dollars invested. How much did he invest? *Ans.*, \$40, or \$60.

15. A man sold a horse for \$312.50, and gained one-tenth as much per cent. as the horse cost him. How much did the horse cost him? *Ans.*, \$250.

Suggestion.—One-tenth the cost is $\frac{x}{10}$, letting x=the cost. Hence his gain was $\frac{x}{1000}$ of the cost. The equation is

$$\frac{x^2}{1000}=312.5-x.$$

16. A set out from C towards D, and traveled 7 miles an hour. After he had gone 32 miles, B set out from D towards C, and went each hour $\frac{1}{19}$ of the whole distance; and after he had traveled as many hours as he went miles in one hour, he met A. Required the distance between the two places. *Ans.*, 152, or 76 miles.

Suggestion.—Letting x = the distance from C to D, $\frac{x}{19}$ = B's distance per hour, and also his time. The equation is

$$32+\frac{7x}{19}+\frac{x^2}{(19)^2}=x.$$

17. A certain number consists of two figures whose sum is 12; and the product of the two figures plus 16 is

equal to the number expressed by the figures in inverse order. What is the number? *Ans.*, 84.

18. Find two numbers whose sum is 8, and the sum of whose cubes is 152.

19. Two travelers, A and B, set out to meet each other, A leaving the town C at the same time that B left D. They traveled the direct road C D, and, on meeting, it appeared that A had traveled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What was the distance between C and D?

Suggestion.—Letting $x=$ number of miles A traveled, the equation is

$$\frac{63x}{4(x-18)}=\frac{28(x-18)}{x},$$

or

$$9x^2=16(x-18)^2,$$

whence

$$3x=4(x-18).$$

20. A merchant bought a piece of cloth for $120, and, after cutting off 4 yards, sold the remainder for what the whole cost him; by which he made $1 a yard on what he sold. How many yards did the piece contain?

SECTION XXXI.

FREEING EQUATIONS OF RADICALS.

1. Given $10+\sqrt{2x+9}=15$ to find the value of x.

SOLUTION.—Transposing and uniting *so that the radical shall constitute one member*, we have

$$\sqrt{2x+9}=5.$$

Squaring each member, which will not destroy the equation, since the squares of equal quantities are equal, we find

$$2x+9=25.$$

Hence $$x=8.$$

2. Given $9+\frac{\sqrt{3x+4}}{5}=10$ to find x. Verify.

3. Given $\frac{3\sqrt{2x+27}}{5}-2\frac{1}{5}=2$ to find x.

Suggestion.—The successive transformed equations are

$$3\sqrt{2x+27}-11=10,$$

$$3\sqrt{2x+27}=21,$$

$$\sqrt{2x+27}=7, \text{ etc.}$$

4. Given $2\sqrt{x-4}=\sqrt{2x}$ to find x. $x=8$.

Suggestion.—Square each member. The square of $2\sqrt{x-4}$ is $4x-16$.

5. Given $\sqrt{7x-13}=\sqrt{91-x}$ to find x. Verify.

6. Given $\sqrt{x-16}=\sqrt{x}-2$ to find x.

Suggestion.—Squaring, $x-16=x-4\sqrt{x}+4$.

Transposing and uniting $4\sqrt{x}=20$.

Dividing by 4, $\sqrt{x}=5$.

Squaring again, $x=25$.

7. Given $2+x=\sqrt{4+x\sqrt{64+x^2}}$ to find x.

Suggestion.—The successive transformed equations are

$$4+4x+x^2=4+x\sqrt{64+x^2},$$
$$4x+x^2=x\sqrt{64+x^2},$$
$$4+x=\sqrt{64+x^2},$$
$$16+8x+x^2=64+x^2,$$
$$8x=48.$$

8. Given $\sqrt{x+13}+\sqrt{x}=13$, to find x. Verify.

9. Given $\sqrt{x-32}=16-\sqrt{x}$, to find x. Verify.

10. Given $\sqrt{x+a}=b+\sqrt{x}$, to find x.
$$x=\frac{(a-b^2)^2}{4b^2}.$$

Find the value of x in the following, which are pure Quadratics after being freed from radicals.

11. $24-\sqrt{2x^2+9}=15$. $x=6$, and -6.

12. $13-\sqrt{3x^2+16}=5$. Verify.

13. $x\sqrt{6+x^2}=1+x^2$. $x=\frac{1}{2}$, and $-\frac{1}{2}$.

14. $\sqrt{a+x}+\sqrt{a-x}=b$. $x=\pm\frac{b}{2}\sqrt{4a-b^2}$.

Suggestion.—Squaring, $a+x+2\sqrt{a^2-x^2}+a-x=b^2$.

Whence $2\sqrt{a^2-x^2}=b^2-2a$.

15. $x\sqrt{a+x^2}=b+x^2$. $x=\pm\frac{b}{a-2b}\sqrt{a-2b}$.

Find the value of the following, which are affected Quadratics after being freed from radicals.

16. $x+5-\sqrt{x+5}=6$. $x=4$, and -1.

Suggestion.—The successive transformed equations are

$$\sqrt{x+5}=x-1$$
$$x+5=x^2-2x+1, \text{ etc.}$$

17. $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$. $x=9$ and -12.

Suggestion. $3\sqrt{x+16}=x+6$.

18. $\sqrt{6-x}=2\sqrt{\dfrac{2}{x}}$. $x=4$, and 2.

19. $\frac{1}{2}\sqrt{x-2}=\sqrt{\dfrac{102-\frac{1}{2}x^2}{x}}$. $x=12$, and $-11\frac{1}{3}$.

20. $\sqrt{x}\times\sqrt{5x-20}=\sqrt{7x-34}$. $x=3\frac{2}{5}$, and 2.

Problems.

1. The ages of two brothers are such that the age of the elder plus the square root of the age of the younger is 22 years, and the sum of their ages is 34 years. What is the age of each? *Ans.*, Elder, 18; younger, 16.

2. A man said that he had sold such a part of his farm that, if he had sold $\frac{1}{3}$ more of it, he would have sold the square root of what he did sell plus its square. What part did he sell? *Ans.*, $\frac{1}{3}$.

The equation is $x+\frac{1}{3}=\sqrt{x+x^2}$.

3. What number is that expressed by two digits which are in the ratio of 1 to 2, and the square root of whose sum is $\frac{1}{12}$ of the number itself?

4. What number is that to which, if 1 be added, the square root of the sum is equal to $\frac{1}{3}$ of the difference between the square root of the number and 1?

The equation is $\sqrt{x+1}=\dfrac{\sqrt{x}-1}{3}$. From which we find x *imaginary*. This means that there is no such number.

5. What number is that to which if 33 be added, the square root of this sum increased by the square root of the difference between the number and 63 is 12?

6. If 4 be subtracted from a father's age, the remainder will be thrice the age of the son; and if 1 be taken from the son's age, half the remainder will be the square root of the father's age. Required the age of each.

Father's, 49; *son's*, 15.

7. A young lady being asked her age, answered, "If you add the square root of my age to $\frac{2}{3}$ of my age, the sum will be 10." Required her age.

8. A stranger asked the distance to a certain place and was told that twice the square root of the distance exceeded 5 miles by twice the reciprocal of the square root of the distance. The stranger replied that this answer was ambiguous. Why was it so?

QUADRATIC EQUATIONS

SIMULTANEOUS, WITH TWO UNKNOWN QUANTITIES.

SECTION XXXII.

One Simple Equation and one Quadratic.

1. Given $2x-5y=11$, and $x^2-y+10=3y^2+149$, to find x and y.

SOLUTION.—Here we have a *Simple Equation* and a *Quadratic*. Putting the equations in their simplest form (the simple equation is in such form), we have

$$2x-5y=11, \qquad \text{and} \qquad x^2-y-3y^2=139.$$

Finding the value of one of the unknown quantities, as x, from the *Simple Equation*, we obtain, $\quad x=\dfrac{11+5y}{2}$.

Substituting this in the *Quadratic*, we have

$$\frac{121+110y+25y^2}{4}-y-3y^2=139.$$

Clearing this of fractions, transposing, and uniting terms,

$$13y^2+106y=435.$$

Solving this *affected Quadratic*, we find

$$y=3, \qquad \text{and} \qquad -\frac{145}{13}.$$

Substituting these values of y in the value of x found from the simple equation, we find that

$$\text{for} \quad y=3, \qquad x=13, \qquad \text{and for} \qquad y=-\frac{145}{13}, \qquad x=-\frac{291}{13}.$$

174. It is very important that the pupil observe how the values of the unknown quantities are related to each other; thus, in this case, $y=3$ and $x=13$ are correlative, and will satisfy the equation. But $y=3$ and $x=-\frac{291}{13}$ will *not* satisfy it; neither will $y=-\frac{145}{13}$ and $x=13$. But $y=-\frac{145}{13}$ and $x=-\frac{291}{13}$ will satisfy the equation.

175. *Two equations between two unknown quantities, one of the first degree and the other of the second, may be solved as a Quadratic by finding the value of one of the unknown quantities in the simple equation, substituting this value in the quadratic, and solving the resulting equation.*

Solve the following:

1. $\left.\begin{array}{l} x+y=9, \\ x^2+y^2=45. \end{array}\right\}$ $x=3,\ y=6$; and $x=6,\ y=3$.

2. $\left.\begin{array}{l} x+2y=7, \\ x^2+3xy-y^2=23. \end{array}\right\}$ $x=3,\ y=2$; and $x=15\frac{2}{3},\ y=-4\frac{1}{3}$.

3. $\left.\begin{array}{l} x-y=-2, \\ x+\frac{y}{10}=\frac{3xy}{10}. \end{array}\right\}$ $x=-\frac{1}{3},\ y=1\frac{2}{3}$; and $x=2,\ y=4$.

4. $\left.\begin{array}{l} 2x+y=10, \\ 2x^2-xy=54-3y^2. \end{array}\right\}$ $x=5\frac{1}{8},\ y=-\frac{1}{4}$; and $x=3,\ y=4$.

176. In solving such equations, care should be taken that the values be given in the proper order. The *First Roots* are those which arise from taking the + sign of the radical in solving the Quadratic.

5. $\left.\begin{array}{l} x-y=5, \\ x^2+y^2=73. \end{array}\right\}$ The values of x are -3 and 8; and of y 3 and -8, which is the *First* pair of values? Which is the second?

6. $\left.\begin{array}{l} x-y=5, \\ xy=36. \end{array}\right\}$ Is the *first* value of x 9, or -4? What is the *corresponding* value of y? What the other values?

7. $\left.\begin{array}{l} 3x^2=24-2y, \\ x-y=\dfrac{1-2y}{5}. \end{array}\right\}$ Verify.

8. $\left.\begin{array}{l} 7x+\frac{1}{4}y=23, \\ xy=42. \end{array}\right\}$ Verify.

9. $\left.\begin{array}{l} ax+by=c, \\ xy=d. \end{array}\right\}$ $x=\dfrac{c\pm\sqrt{c^2-4abd}}{2a}$. $y=\dfrac{c\mp\sqrt{c^2-4abd}}{2b}$.

Two Homogeneous Quadratics.

177. A Homogeneous Equation *is an equation in which each term into which the unknown quantities enter has the same number of unknown factors.*

Thus $xy=8$, and $x^2-2xy+y^2=4$ are homogeneous; but $2x+y=7$, and $x^2+3xy-y^2=23$, are not homogeneous *with each other*, though each is homogeneous in itself. $5x-2xy=10$ is not homogeneous, nor is $x^2-xy+y=81$.

Solve $xy=8$, and $x^2-2xy+y^2=4$.

Solution.—Let $y=vx$, v being an unknown multiplier which it will be our purpose to determine. Substituting in both equations vx for y, the first becomes $vx^2=8,$ (3)

and the second, $x^2-2vx^2+v^2x^2=4.$ (4).

From (3) $x^2=\frac{8}{v}$; and from (4), $x^2=\frac{4}{1-2v+v^2}$.

Placing these values of x^2 equal to each other, since the equations are supposed simultaneous,

$$\frac{8}{v}=\frac{4}{1-2v+v^2}, \text{ or } \frac{2}{v}=\frac{1}{1-2v+v^2}.$$

Solving the last we find

$$2-4v+2v^2=v,$$
$$2v^2-5v=-2.$$

Whence $v=2$, and $\frac{1}{2}$.

Taking the former, (3) becomes $2x^2=8$;

whence $x=\pm 2$, and $y=vx$,

gives $y=\pm 4.$

The value $v=\frac{1}{2}$ would simply exchange the values of x and y, making $x=\pm 4$, and $y=\pm 2$.

178. Two Homogeneous Quadratic Equations *between two unknown quantities can always be solved by the method of quadratics, by substituting for one of the unknown quantities the product of a new unknown quantity into the other.**

Solve the following by the above method, being sure to observe which values of x and y go together. The numerical values will be written in the margin without indication as to which of the unknown quantities they are the values of, or as to their order:

1. $\left.\begin{aligned} x^2+xy &=60, \\ xy+2y^2 &=133. \end{aligned}\right\}$ $5, -5, 12\sqrt{2}, -12\sqrt{2}, \frac{12}{2}\sqrt{2}, -\frac{12}{2}\sqrt{2}, -7, +7.$

* Of course such equations can be solved without this expedient, and some of them by more elegant methods, but this form of solution is of great practical importance and should be familiar.

2. $\left.\begin{array}{l} xy-5y^2=-102, \\ x^2-2y^2=97. \end{array}\right\}$ 6, 13, -6, -13, omitting the negative value of v.

3. $\left.\begin{array}{l} 2x^2-3xy=56, \\ xy-y^2=15. \end{array}\right\}$ ± 3, ± 8, omitting the second value of v.

4. $\left.\begin{array}{l} 2x^2-xy=6, \\ 2y^2+3xy=8. \end{array}\right\}$ $+2$, -2, $+\frac{8}{7}\sqrt{7}$, $-\frac{8}{7}\sqrt{7}$, $+\frac{3}{7}\sqrt{7}$, $-\frac{3}{7}\sqrt{7}$, $+1$, -1.

5. $\left.\begin{array}{l} 5x^2-3xy=56, \\ 5y^2+xy=28. \end{array}\right\}$

6. $\left.\begin{array}{l} 4x^2=3xy-2, \\ x^2+y^2=5. \end{array}\right\}$

7. $\left.\begin{array}{l} x^2+xy=12, \\ xy-2y^2=1. \end{array}\right\}$

8. $\left.\begin{array}{l} 3x^2-3xy+y^2=21, \\ 2xy=3y^2+x^2-19. \end{array}\right\}$

Problems.

1. If a certain number, consisting of two places, be divided by the product of its digits the quotient will be 2, and if 27 be added to it, the digits will be in an inverted order; required the number. *Ans.*, 36.

2. The *perimeter*, or sum of the four sides of a rectangle, is 112 rods, and its area is 720 square rods. What are the length and breadth of the rectangle?

Ans., 36, and 20 rods.

3. The fore wheels of a carriage make 2 revolutions

more than the hind wheels in going 90 yards; but if the circumference of each wheel is increased 3 feet, the carriage must pass over 132 yards in order that the fore wheels may make 2 revolutions more than the hind wheels. What is the circumference of each wheel?

Ans., Fore wheels, $13\frac{1}{2}$ feet; hind wheels, 15 feet.

4. A and B start at the same time, from two different points, and travel towards each other; when they meet on the road, it appears, that A has traveled 30 miles more than B. It also appears, that it will take A 4 days to travel the road that B has come, and B 9 days to travel the road that A has come. Find the distance of A from B, when they set out. *Ans.*, 150 miles.

5. Two persons, A and B, depart from the same place, and travel in the same direction; A starts 2 hours before B, and after traveling 30 miles, B overtakes A; but had each of them traveled half a mile more per hour, B would have traveled 42 miles before overtaking A. At what rate did they travel?

Ans., A $2\frac{1}{2}$, and B 3 miles per hour.

6. The area of a rectangular field is 1575 square rods; and if the length and breadth were each lessened 5 rods, its area would be 1200 square rods. What are the length and breadth?

7. A man had a field 4 times whose length equaled 6 times its breadth. He gave 3 dollars a rod to have it fenced; and the whole number of dollars was equal to the number of square rods in the field. Required the length and breadth of the field.

8. What two numbers are those whose difference is 8, and the sum of whose squares is 544?

9. Divide the number 100 into two such parts that the sum of their square roots may be 14.

Suggestion.—Let x^2 and y^2 be the numbers.

10. The product of two numbers is 12, and the sum of their cubes 91. What are the numbers?

11. Divide 18 into two parts such that the square of the first divided by the second, plus the square of the second divided by the first, shall be 27.

12. The product of two numbers is a, and their quotient is b. What are the numbers?

Ans., $\sqrt{ab}$, and $\sqrt{\frac{a}{b}}$.

13. The sum of the squares of two numbers is a, and the difference of their squares is b. What are the numbers?

Ans., $\sqrt{\frac{a+b}{2}}$, and $\sqrt{\frac{a-b}{2}}$.

14. Divide a into two such parts that the sum of their square roots, shall be b.

The parts are

$$\frac{a+b\sqrt{2a-b^2}}{2}, \text{ and } \frac{a-b\sqrt{2a-b^2}}{2}.$$

ARITHMETICAL PROGRESSION

SECTION XXXIII.

179. **An Arithmetical Progression** *is a series of numbers which increase or decrease by a common difference,* as 3, 5, 7, 9, 11; or 28, 23, 18, 13, 8.

180. A Progression is *Increasing* or *Decreasing* according as the terms increase or decrease in passing to the right. The terms *Ascending* and *Descending* are used in the same sense as increasing and decreasing, respectively. In an increasing Arithmetical Progression the common difference is added to any one term to produce the next term to the right; and in a decreasing progression it is subtracted. An A. P. is indicated thus, 3..5..7..9..11, etc.

181. There are *Five Things* to be considered in any progression; viz., the first term, the last term, the common difference or the ratio, the number of terms, and the sum of the series, either three of which being given the other two can be found, as will appear from the subsequent discussion.

1. Write 10 terms of the increasing Arithmetical Progression whose first term is 3 and common difference 5. Seven terms of one whose first term is 1 and common difference 2.

2. Write 8 terms of the decreasing A. P. whose first term is 123 and common difference 4. Five terms of one whose first term is 343 and common difference 20.

Queries.—In an ascending A. P., how many times must the C. D. (Common Difference) be added to the first term to produce the 5th? The 7th? The 15th? In a decreasing A. P., how many times must the C. D. be subtracted from the first term to produce the 6th term? The 11th? The 20th?

3. Write 10 terms of an increasing A. P. whose first

term is a and C. D. d. What is the 10th term? What would be the 13th? The 24th?

182. **The Last Term** *of an* **Increasing** *arithmetical progression is equal to the first term + the common difference taken as many times as there are terms less 1.*

Thus the 5th term of the 1st series above is 3 + 4 times 2 = 11.

The last term of a **Decreasing** *arithmetical progression is equal to the first term – the common difference multiplied by the number of terms less 1.*

Thus the 5th term of the second series above is 28 — 4 times 5 = 8.

4. If a is the first term, d the C. D. and n the number of terms, and l the last term, show that

$$\mathbf{l = a + (n - 1) d}. \qquad (1.)$$

When the progression is decreasing, we have simply to regard **d** *as minus.*

5. Apply formula (1) to find the last term of the increasing A. P., when $a = 7$, $n = 42$, and $d = 11$. When $a = 13$, $n = 150$, and $d = 17$.

6. Apply formula (1) to find the last term of a decreasing A. P. when $a = 247$, $n = 13$, and $d = 15$. When $a = 437$, $n = 21$, and $d = -17$. When $a = 28$, $d = -12$, and $n = 8$.

In the last case we have $l = 28 - 7 \times 12 = 28 - 84 = -56$.

7. If the first term is 2, the last term 32, and the C. D. 3, what is the number of terms?

SOLUTION.—Substituting in the formula $l = a + (n - 1)d$, we have $32 = 2 + (n - 1)3$. Whence, solving for n, we have $n = 11$.

8. A man going a journey traveled the first day 7 miles, the last day 67 miles, and each day increased his journey by 4 miles. How many days did he travel?

9. A man traveled 16 days, going 4 miles the first day and 67 the last. What was his daily increase of rate, supposing it uniform?

10.* A certain river is 150 miles long; its source is $927\frac{1}{2}$ *ft.* above the level of the sea, and its mouth $421\frac{2}{3}$ *ft.* What is the average fall per mile? *Ans.* 3.4 *ft.*—.

183. *The first and last terms are called the* **Extremes** *of a progression, and the others the* **Means.**

11. Insert 4 arithmetical means between 17 and 57.

As there are 2 extremes and 4 means in the required series, there are 6 terms. Hence we know the first term, (17), the last term, (57), and the number of terms, (6), and can find the C. D. The means are 25, 33, 41, 49.

12. What is the arithmetical mean between 5 and 7? 23 and 11? 17 and 22?

184. *The formula for inserting a given number of arithmetical means between two given extremes is*

$$d = \frac{l - a}{m + 1},$$

in which m *represents the number of means. From*

* If teachers need more examples for class drill than are here supplied, they will find the Author's "TEST EXAMPLES IN ALGEBRA" a satisfactory resort.

this **d**, *the common difference. being found, the terms can readily be written.*

DEM.—If a is the first term and l the last, and there are m terms between, or m means, there are in all $m + 2$ terms. Hence, substituting in the formula $l = a + (n - 1)d$, for n, $m + 2$, we have $l = a + (m + 1)d$. From this $d = \frac{l - a}{m + 1}$.

13. Insert 10 arithmetical means between 127 and 3. 7 between 155 and 128. 2 between 3 and 4. 5 between 58 and 88.

14. A wishes to place 26 trees in front of a lot 300 *ft.* on the street, placing the trees at equal distances from each other, and one at each corner. How far apart must he place them?

This is a very simple question, but the purpose is to have the pupil apply the formula $l = a + (n - 1)d$ to its solution.

15. Show that $2n - 1$ represents the nth *odd* number, and $2n$ the nth *even* number.

---

185. The formula for the sum of an Arithmetical Progression *is*

$$s = \left[\frac{a + l}{2}\right]n,$$

s *representing the sum of the series,* **a** *the first term,* **l** *the last term, and* **n** *the number of terms.*

DEM.—If l is the last term of the progression, the term before it is $l - d$, and the one before that $l - 2d$, etc. Hence, as $a \ldots a+d \ldots a+2d \ldots a+3d$ - - - - - - l represents the series, $l \ldots l - d \ldots l - 2d \ldots l-3d$ - - - - - a represents the same series reversed. Now the sum of the first series is

$$\begin{array}{ll} & s = a+(a+d)+(a+2d)+ \cdots (l-2d)+(l-d)+l; \\ \text{and reversed} & s = l+(l-d)+(l-2d)+ \cdots (a+2d)+(a+d)+a. \\ \hline \text{Adding} & 2s = (a+l)+(a+l)+(a+l)+ \cdots (a+l)+(a+l)+(a+l). \end{array}$$

If the number of terms in the series is n, there will be n terms in this sum, each of which is $(a+l)$; hence $2s = (a+l)\,n$, or

$$s = \left[\frac{a+l}{2}\right] n.$$

1. If the first term is 5, the last term 23, and the number of terms 7, what is the sum of the series?

2. What is the sum of the first 13 *odd* numbers? Of the first 13 *even* numbers?

3. Show that the sum of the first n odd numbers is n^2. Of the first n even numbers is n^2+n.

4. Apply the above to find the sum of the first 17 odd numbers. The first 20 even numbers.

5. Show that $\frac{1}{2}(n^2+n)$ is the formula for the sum of the first n natural numbers, and hence that this sum is $\frac{1}{2}$ the sum of the first n even numbers.

6. Show that the common clock strikes 156 strokes in 24 hours.

7. A 10% note for \$300, bearing annual interest, has been running 8 *yr.* and no interest has been paid. What is due, allowing simple interest on the deferred payments of annual interest?

The 8th year's interest is \$30, the 7th \$33, the 6th \$36, etc. Hence the interest is an arithmetical progression of 8 terms of which 30 is the first and 3 the com. diff. The last term is therefore 51, and the sum $\left(\frac{51+30}{2}\right) \times 8 = 324$. Amount of note \$624.

8. Show that on such a note as the above the interest on the interest is $\frac{1}{2}(n^2-n)\,r^2p$, n being the number of

years, r the rate (represented as 100ths), and p the principal; and hence that $p+nrp+\frac{1}{2}(n^2-n)r^2p$, or

$$p\{1+nr[1+\tfrac{1}{2}(n-1)r]\}$$

is the formula for the amount of such a note.

Apply the formula $p\{1+nr[1+\frac{1}{2}(n-1)r]\}$ to the two following:

9. On a note for $150 bearing annual interest at 7%, the debtor had neglected to pay the interest for 3 *yr.* Allowing simple interest on the deferred payments, what was then due on the note? *Ans.*, $183.71.

10. On the same principle as in the last, what is due on a note for $525, bearing 6% annual interest, the interest payments having been deferred 4 years?

Ans., $662.34.

11. A body falls $16\frac{1}{12}$ feet the 1st second, 3 times as far the 2d second, 5 times as far the 3d second, and so on; how far does it fall in 20 seconds? *Ans.*, $6433\frac{1}{3}$ *ft.*

12. If a body fall n feet the 1st second, $3n$ feet the 2d, $5n$ feet the 3d, and so on, how far will it fall in t seconds?

Ans., t^2n feet.

---

186. *Formulas*

$$l = a + (n-1)d, \tag{1}$$

and

$$s = \left[\frac{a+l}{2}\right]n, \tag{2}$$

being two equations between the five quantities a, l, n, d, *and* s, *any three of these being given, the other two can be found.*

1. If $a = 3$, $s = 465$, and $d = 4$, find n and l.

2. If $a = 3$, $l = 59$, and $d = 4$, find n and s.

3. If $n = 15$, $l = 59$, and $s = 465$, find a and d.
4. If $a = 3$, $n = 15$, and $s = 465$, find d and l.
5. If $a = 3$, $n = 15$, and $l = 59$, find d and s.
6. If $l = 59$, $d = 4$, and $s = 465$, find a and n.
7. If $a = 3$, $l = 59$, and $s = 465$, find n and d.
8. If $d = 4$, $n = 15$, and $s = 465$, find a and l.
9. It $l = 59$, $d = 4$, and $n = 15$, find a and s.

GEOMETRICAL PROGRESSION

SECTION XXXIV.

187. A Geometrical Progression *is a series of numbers which increase or decrease by a common multiplier, called the* **Rate.** *If the rate is more than 1 the series is increasing; if less than 1 it is decreasing.*

Thus 3, 9, 27, 81, 243, is an increasing geometrical progression, rate 3. 6561, 729, 81, 9, is a decreasing geometrical progression, rate $\frac{1}{9}$.

1. Write six terms of a geometrical series whose first term is 2 and rate (often called ratio) 3.

2. Write 5 terms of a G. P. whose first term is 1 and rate $\frac{1}{2}$. One whose first term is 5 and rate 3.

3. Write 8 terms of a G. P. whose first term is a and rate r.

Query.—Does it make any difference in the form of the last progression whether r is integral or fractional?

188. The sign : is used to indicate that numbers are in a G. P Thus, 5 : 10 : : 20 : 40, etc. The terms ascending, descending, extremes, and means, are used in a similar way in both arithmetical and geometrical progressions.

189. *The Formula for finding the* **nth** *or* **Last Term** *of a G. P. is*

$$l = ar^{n-1}.$$

in which **l** stands for last, or **n**th term, **a** the first term, **r** the rate, and **n** the number of terms.

DEM.—Letting a represent the first term and r the ratio, the series is $a : ar : ar^2 : ar^3 : ar^4$: etc. Whence it appears that any term consists of the first term multiplied into the ratio raised to a power whose exponent is one less than the number of the term. Therefore the nth term, or $l = ar^{n-1}$.

4. What is the 13th term of 3 : 6 : 12, etc.? Of 164 : 82 : 41, etc.?

5. What is the 15th term of $\frac{1}{2} : \frac{1}{4} : \frac{1}{8}$, etc. Of $1 : \frac{1}{12} : \frac{1}{144}$, etc.

---

190. *The Formula for the* **Sum** *of a* **Geometrical Series** *is*

$$s = \frac{ar^n - a}{r - 1},$$

in which **s** represents the sum, **a** the first term, **r** the ratio and **n** the number of terms.

DEM.—The sum of the series being found by adding all its terms, we have

$s=a+ar+ar^2+ar^3+\cdots ar^{n-3}+ar^{n-2}+ar^{n-1}$, and multiplying by r, $rs=ar+ar^2+ar^3+\cdots ar^{n-3}+ar^{n-2}+ar^{n-1}+ar^n$.

Subtracting, $$rs-s=ar^n-a,$$

or $$(r-1)s=ar^n-a,$$

and $$s=\frac{ar^n-a}{r-1}.$$

SUG.—The student will notice that multiplying the first series by r, and placing the terms of the product under the like terms of the series, simply moves each term, when multiplied, one place to the right, so that however many terms there may be in the series, each will have a similar one in the product except the first term, a; and each term in the product will have a similar one in the series, except the last one, ar^n

1. What is the sum of 8 terms of the series 3 : 6 : 12, etc.? Of 5 : 15 : 45, etc.

2. What is the sum of 5 terms of $\frac{1}{2}:\frac{1}{6}:\frac{1}{18}$, etc.? Of 7 terms of $4:\frac{4}{3}:\frac{4}{9}$, etc.?

3. There being 8 nails in each shoe, what would the shoeing of a horse "all round" cost at 1¢ for the first nail, 2¢ for the second, 4 for the 3d, etc.?

Ans. \$42,949,672.95.

4. What would the shoeing of one foot cost at the above rate? What of two feet? *Ans.* \$2.55; \$655.35.

---

191. *Any term of a G. P. is a* **Geometrical Mean** *between the two adjacent terms.*

The term is usually applied when there are but three terms considered.

5. Show that a geometrical mean between two numbers is the square root of their product.

6. What is the geometrical mean between 4 and 9? Between 4 and 16? Between 3 and 7?

7. Show that the general formula for inserting m geometrical means between l and a is $r = \sqrt[m+1]{\frac{l}{a}}$, r being the rate.

SOLUTION.—Since there are to be m means and 2 extremes, there will be $m + 2$ terms in all. Now $l = ar^{n-1}$; and as n in this case is $m + 2$, we have $l = ar^{m+1}$; whence $r = \sqrt[m+1]{\frac{l}{a}}$.

8. Insert 5 geometrical means between 7 and 5203.

Suggestion.—The ratio is the 6th root of $\frac{5203}{7}$, or the cube root of the square root of 729.

---

192. *The formula for the sum of an* **Infinite Decreasing Geometrical Progression** *is*

$$s = \frac{a}{1 - r}.$$

DEM.—Since the terms are growing less and less, when the series is extended to infinity l becomes 0; whence the formula $s = \frac{a - lr}{1 - r}$ becomes $s = \frac{a}{1 - r}$.

1. What is the sum of $1 : \frac{1}{2} : \frac{1}{4}$, etc., to infinity, *i. e.*, extended forever? *Ans.*, 2.

2. What is $\frac{1}{2} + \frac{3}{8} + \frac{9}{32}$, etc., to infinity?

3. What is $14 + 2 + \frac{2}{7}$, etc., to infinity?

4. What is the value of $\dot{3}\dot{2}$?*

The repetend $.\dot{3}\dot{2}$ is $\frac{32}{100} + \frac{32}{10000} + \frac{32}{1000000}$, etc., *i. e.*, an infinite decreasing geometrical progression, in which the first term is $\frac{32}{100}$, and the rate $\frac{1}{100}$. Hence $s = \frac{\frac{32}{100}}{1-\frac{1}{100}} = \frac{32}{99}$.

5. What is the value of $.\dot{2}$? Of $.0\dot{3}$? Of $.\dot{0}\dot{3}$? Of $.0\dot{2}\dot{5}$? Of $.\dot{0}2\dot{5}$?

Problems involving Progressions.

1. Sum to n terms the A. P. whose rth term is $2r - 1$.

Since r is general and applies to any term where $r = 1$, we have $2r - 1 = 1$, the *first* term. When $r = 2$, we have $2r - 1 = 3$, the *second* term. When $r = 3$. we have $2r - 1 = 5$, the *third* term. The nth term is $2n - 1$. Hence the series is 1, 3, 5, - - - to $2n - 1$. Therefore $s = \left(\frac{l+a}{2}\right) n = \left(\frac{2n-1+1}{2}\right) n = n^2$.

2. Find the G. P. in which the sum of n terms is $\frac{1}{2}(3^n - 1)$

3. What is the G. P. whose nth term is s^n.

4. There is a number consisting of three digits, which are in arithmetical progression; the quotient arising from dividing the number by the sum of its digits is 26; but if 198 be added to the number, its digits will be reversed. What is the number? *Ans.* 234.

---

* This means .323232 to infinity, and is called a *Repetend.* $\dot{2}$ = .2222 to infinity, or $\frac{2}{10}+\frac{2}{100}+\frac{2}{1000}$ to infinity. $.\dot{0}\dot{3}$ = .030303 to infinity, or $\frac{3}{100} + \frac{3}{10000} + \frac{3}{1000000}$ to infinity. $.0\dot{3}$ = 03333 to infinity, or $\frac{1}{10}$ of .333 to infinity, or $\frac{1}{10}$ of the sum of the series $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}$ to infinity.

Let x represent the middle digit and y the C. D. Then the digits are $x-y$, x, and $x+y$, and the number is $100(x-y) + 10x + x+y$, or $111x - 99y$. The equations are $\frac{111x-99y}{3x} = 26$, and $111x - 99y + 198 = 100(x+y)+10x+x-y = 111x+99y$.

5. A body near the earth falling by its own weight, if it were not resisted by the air, would descend in the first second of time through a space of 16 feet and 1 inch; in the next second through 3 times that space; in the third, through 5 times that space; in the fourth, through 7 times that space, etc. Through what space would it fall at the same rate of increase in a minute?

Ans., 57900 feet.

6. Find three numbers in geometrical progression, whose sum shall be 52, and the sum of the extremes to the mean, as 10 to 3.

Let x be the first number and y the rate, so that the numbers shall be x, xy, and xy^2.

7. There are three numbers in geometrical progression whose sum is 13, and the sum of whose extremes multiplied by the mean is 30. Find the numbers.

Let $\frac{x}{y}$, x, and xy represent the numbers. Then $\frac{x}{y}+x+xy = 13$, and $x\left(\frac{x}{y}+xy\right) = 30$. Whence $\frac{x}{y}+xy = \frac{30}{x}$, and $x + \frac{30}{x} = 13$, etc.

8. What kind of a series is x, $\sqrt{xy}$, y? What is the rate?

9. What kind of a series is $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$? What is the rate?

10. The sum of three numbers in geometrical progression is 39, and the sum of their squares 819. What are the numbers?

Represent the numbers as in Ex. 8.

11. There are four numbers in geometrical progression whose continued product is 64; and the sum of the series is to the sum of the means as 5 : 2. What are the numbers? *Ans.*, 1, 2, 4, 8.

Representing the numbers as in Ex. 9, we have $\frac{x^2}{y}\times x\times y\times\frac{y^2}{x} = x^2y^2 = 64$, whence $xy = 8$; and $\frac{x^2}{y}+x+y+\frac{y^2}{x} : x+y :: 5 : 2$, $\therefore \frac{x^2}{y}+\frac{y^2}{x} : x+y :: 3 : 2$; and $\frac{x^3+y^3}{8} : x+y :: 3 : 2$, $\frac{x^3+y^3}{x+y} : 8 :: 3 ; 2$, $x^2-xy+y^2 : 8 :: 3 : 2$, $x^2-8+y^2 : 8 :: 3 : 2$, $x^2+y^2 : 8 :: 5 : 2$. Hence we have to solve $xy = 8$, and $x^2+y^2 = 20$. From these $x^2+2xy+y^2 = 36$, or $x+y = 6$; and $x^2-2xy+y^2 = 4$, or $x-y = 2$.

SECTION XXXV.

PERCENTAGE FORMULAS.

193. *The* **Fundamental Formulas** *of Percentage are*

$$\mathbf{p} = \mathbf{br}, \tag{1}$$

and

$$\mathbf{A} = \mathbf{b} + \mathbf{br}, \tag{2}$$

or

$$\mathbf{A} = \mathbf{b}(1 + \mathbf{r}),$$

in which **b** represents the *base*, **r** the *rate* (represented in hundredths), **p** the *percentage*, and **A** the *amount*.*

---

* The pupil is presumed to be familiar with Elementary Arithmetic.

To produce $p = br$, we have but to remember the definitions of *base*, *rate* and *percentage*. This is illustrated by the problem, To take 8% of 345. Now 8% of anything means .08 of it; hence 8% of 345 is $345 \times .08$, and the *percentage* $= 345 \times .08$. In the formula $p = br$, r is hundredths.

To produce the second formula we have but to remember the definition of *amount*. From this, $A = b + br$, since b is the base and br the percentage, and the sum of these two is the amount.

194. *The* **Fundamental Formulas** *in simple interest are*

$$\mathbf{I = Prt,} \qquad (1)$$

and $$\mathbf{A = P + I = P + Prt = P(1 + rt),} \qquad (2)$$

in which **P** is the *Principal,* or base; **r**, the *Rate;* **t**, the *Time* in years; **i**, the *Interest,* or percentage; and **A** the *Amount.*

DEMONSTRATION. 1. When a principal P is put at simple interest at rate r, for t years, the understanding is that the interest for one year is r times the principal, or Pr, and that for 2 years it is *twice* as much as for 1 year; for 3 *yr.*, 3 times as much; for $3\frac{1}{2}$ or $3\frac{1}{4}$ years, $3\frac{1}{2}$ or $3\frac{1}{4}$ times as much as for 1 *yr.* Hence the interest for 1 *yr.* being Pr, for t years it is t times as much, or Prt, and we have $i = Prt$.

2. To produce formula (2) we have but to remember that A, the amount, is the sum of principal and interest. Hence $A = P + i$, by definition. But as by (1) $i = Prt$, we have also $A = P + Prt$, and as $P + Prt$ is the same as $P(1 + rt)$, we have

$$A = P + i = P + Prt = P(1 + rt).$$

1. Selling \$250 worth of goods at a profit of 12%, how much is gained? For what are the goods sold?

Formula 1, (193), gives $p = br = \$250 \times .12 = \$30$.
Formula 2, (193), gives $A = \$250(1 + .12) = \$250 \times 1.12 = \$280$.

2. Selling at a profit of 12%, I make \$30. How much is sold? How much is received?

From $p = br$, we have $\$30 = b \times .12$, $\therefore b = \frac{\$30}{.12} = \$250$.

From $A = b(1+r)$, we have $A = \$250 \times 1.12 = \$280$.

3. Selling \$250 worth of goods for \$280, what per cent is made?

$A = b(1+r)$, gives $\$280 = \$250(1+r)$. $\therefore r = .12$, and the rate per cent is 12.

---

4. How long does it take \$480 to gain \$450, at 15% simple interest?

Formula $i = prt$, gives $450 = 480 \times .15t$. Hence $t = \frac{450}{480 \times .15} =$ (by cancelling) $\frac{25}{4} = 6\frac{1}{4}$. $6\frac{1}{4}$ *yr.* = 6 *yr.* 3 *mo.*

5. At what per cent will \$480 gain \$450 interest, in 6 *yr.* 3 *mo.* ($6\frac{1}{4}$ *yr.*)?

From $i = prt$, we have $r = \frac{i}{pt} = \frac{450}{480 \times 6\frac{1}{4}} = .15$. $\therefore 15\%$.

---

6. What is the present worth* of \$793.60, due 3 *yr.* 6 *mo.* hence, without interest, money being worth 8%?

Formula $A = p(1+rt)$, gives $p = \frac{A}{1+rt} = \frac{793.60}{1+(.08 \times 3\frac{1}{2})} = 620$.

195. *The formula for* **Common Discount** *is*

$$p = \frac{A}{1 + rt},$$

in which **p** stands for the discount.

---

* The *Present Worth* is a sum which, put at interest for the given rate and time, produces the given amount

Solve the following by applying the formulas, verifying them by the common arithmetical processes:

7. At what rate per cent does $50 principal yield $5.25 in 1 *yr.* 9 *mo.*

8. In what time will $419.84, yield $41.28 interest, at 5% ?

9. What principal amounts to $232.50, in 2 *yr.* 11 *mo.*, at 10% ?

10. What is the present worth of a debt of $1860 due 4 *yr.* 9 *mo.* 18 *da.* hence, without interest, money being worth 5% ? *Ans.* $1500.

11. If by selling coffee at 27¢ per pound there is a gain of 20%, what must be the selling price to give a gain of $16\frac{2}{3}$% ?

12. My agent in New York bought a bill of goods for me amounting to $3500. What must I remit, allowing him a commission of $2\frac{1}{2}$% ?

13. Bought cloth for $1.50 a yard; how much will be the loss per cent if I sell it at $1.25 a yard ?

14. A sold pork for $87\frac{1}{2}$% of its cost, and thereby lost 3.33\frac{1}{3}$ on a barrel; required the cost per barrel.

15. What is the amount of $746.25 for 1 *yr.* 10 *mo.* and 12 *da.*, at 5% ?

16. At what rate per cent will $750 yield 224.33\frac{1}{3}$ interest in 3 *yr.* 8 *mo.* 26 *da.* ?

17. In what time will $750 yield 224.33\frac{1}{3}$ interest at 8% ?

18. What principal will yield \$224.33$\frac{1}{3}$ interest in 3 *yr.* 8 *mo.* 26 *da.*, at 8%?

19. If I sell $\frac{3}{4}$ of an article for what $\frac{1}{3}$ of it cost me, what % do I lose on the part sold?

20. If $\frac{4}{5}$ of the buying price equals the selling price, what is the loss per cent?

21. If $\frac{4}{5}$ of the selling price equals the buying price, what is the gain per cent?

Partnership

Principle.

196. *In Simple Partnership, i. e., when all the capital is employed the same length of time, the fundamental principle is that the shares of the gains shall bear the same ratio as the shares of the stock.*

1. A, B, and C entered into partnership. A put in \$240, B put in \$400, and C put in \$320. They gain \$192. How much is each man's gain?

Let x represent A's gain. Then \$960 (the whole stock) : \$240 (A's stock) : : \$192 (the whole gain) : x (A's gain). The proportion 960 : 400 : : 192 : y, gives B's gain, and 960 : 320 : : 192 : z, C's.

2. A, B, and C enter into partnership. A puts in \$360, B puts in \$440, and C puts in \$500. They gain \$780. How much is each man's gain?

3. A, B, C, and D hired a pasture for \$12; A put in 12 sheep, B put in 16, C 18, and D 14. How much ought each to pay?

4. Four men traded in company and gained \$1680; A's stock was \$2000, B's \$1600, C's \$2400, and D's \$2000. How much is each man's gain?

5. A farm was purchased for \$7000, by A, B, and C; A furnished \$2500, B \$3000, and C \$1500. They receive \$560 rent yearly. How much of this rent should each receive?

6. Divide \$960 among three persons in such a manner that their shares shall be to each other as 5, 4, and 3 respectively.

7. Two persons form a partnership in trade, with a capital of \$1500, of which the first contributed \$940, and the second the remainder. They gain \$640. How much is each one's share?

8. Divide the number 230 into three parts which shall be to one another as $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$.

Principle.

197. *In Compound Partnership,* i. e., *partnership in which the several partners' shares of the capital are in for different lengths of time, the gain or loss is divided in the ratio of the products of the several amounts of stock into the times which they respectively remained in the business. This is assuming that the use of* \$a *for time* t *in business is equal to* \$at *for time* 1.

1. Three partners, A, B, and C put money into trade as follows: A put in \$50 for 4 months, B \$150 for 2 months, and C \$250 for 3 months. They gained \$250. How much is each man's share of the gain?

$$(50 \times 4) + (150 \times 2) + (250 \times 3) : 50 \times 4 :: 250 : x,$$

or $$1250 : 200 :: 250 : x,$$

gives $$x = \frac{250 \times 200}{1250} = 40, \text{ as A's share.}$$

In like manner, $$1250 : 150 \times 2 :: 250 : y,$$

gives $$y = 60, \text{ B's share.}$$

So also, $$1250 : 250 \times 3 :: 250 : z,$$

gives $$z = 150, \text{ C's share.}$$

2. A, B, and C hired a pasture for $240; A put in 16 cows for 10 weeks, B 20 cows for 7 weeks, and C 25 cows for 6 weeks. How much ought each to pay?

3. A, B, C, and D have together performed a piece of work, for which they receive $266.40. A worked 16 days of 10 hours each, B worked 20 days of 12 hours each, C worked 14 days of 10 hours each, and D worked 15 days of 12 hours each. How much should each receive?

Alligation

ALTERNATE.

198. Alligation Alternate *teaches the method of finding how much of several ingredients, the values of which are known, must be taken to make a compound of a certain value.*

1. How much wine, at 110¢ per gallon, 60¢ per gallon, and 40¢ per gallon, must be mixed together, so that the mixture may be worth 80¢ per gallon?

x, y, and z represent the number of gallons of each kind respectively.

We then have $$110x + 60y + 40z = (x + y + z)\,80,$$

whence $$30x = 20y + 40z.$$

Now, as we have three unknown quantities and but one equation, we may assign any values we please to two of them, which will give positive values to the other, and find a corresponding value to the other. Thus, y and z may each be 1; whence, $x = 2$. Or y may be 2 and z 1; whence $x = \frac{80}{30} = 2\frac{2}{3}$, etc.

2. How much sugar, at 5, 8, and 10 cents a pound must be mixed with 64 pounds, at 12 cents a pound, so that the mixture may be worth 9 cents a pound?

3. How many bushels of oats at 40¢ per bushel, barley at 45¢, and corn at 75¢, must be mixed with 60 *bu.* of rye at 80¢, so that a bushel of the mixture may be worth 60¢?

Let x = the number of bushels of oats, y of barley, and z of corn, then the statement is

$$40x + 45y + 75z + 80 \times 60 = 60\,(x + y + z + 60),$$

or,

$$15z = 20x + 15y - 1200.$$

In this we may give any value we please to y or x, and any value to the other which will make the second member positive, and find a corresponding value to z. Thus if we make $x = 40$, and $y = 50$,

$$z = \frac{800 + 750 - 1200}{15} = \frac{350}{15} = 23\tfrac{1}{3}.$$

Again make $x = 60$, and $y = 30$; whence

$$z = \frac{1200 + 450 - 1200}{15} = 30, \text{ etc.}$$

4. A merchant has sugar worth 5, 6, 9, and 12 cents a pound; with a mixture of these he wishes to fill a hogshead that shall contain 220 pounds. How much of each kind must he take, so that the compound may be worth 8 cents a pound?

Letting x, y, z, and w be the number of pounds of each kind respectively, we have

$$5x + 6y + 9z + 12w = 8 \times 220 = 1760.$$

In this we may give any values we please to any three which will not make the other negative, and find a corresponding value for the other.

PROBLEMS FOR REVIEW

OR TEST QUESTIONS.

SECTION XXXVI.

1. Given $(a+x)(b+x)-a(b+bc) = a^2c+x^2$, to find the value of x. $= ac.$

2. Given $\frac{4x+3}{6x-43} : 1 :: 2x+19 : 3x-19$, to find the value of x. $x = 8.$

3. Given $\sqrt{4x+21} = 2\sqrt{x}+1$, to find the value of x. $x = 25.$

4. Given $\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}} = \frac{a}{b}$, to find the value of x.

$$x = \left(\frac{a+b}{a-b}\right)^2 b.$$

5. Given $\sqrt{x}+\sqrt{x-24} = \frac{60}{\sqrt{x-24}}$, to find the value of x. $x = 49.$

6. Given $\sqrt{1+x\sqrt{x^2+12}} = 1+x$, to find the value of x. $x = 2.$

7. Given $\frac{4x-17}{9} - \frac{3\frac{2}{3}-22x}{33} = x - \frac{6}{x}\left\{1-\frac{x^2}{54}\right\}$, to find the value of x. $x = 3.$

8. Given $\left\{\begin{array}{l} 2x+3y=18, \\ 3x-2y=1, \end{array}\right\}$ to find the values of x and y. $x=3,\ y=4.$

9. Given $\left\{\begin{array}{l} 2x-\frac{y-3}{5}=4, \\ 3y+\frac{x-2}{3}=9, \end{array}\right\}$ to find the values of x and y. $x=2,\ y=3.$

10. Given $\left\{\begin{aligned} x-\frac{y-2}{7}&=5, \\ 4y-\frac{x+10}{3}&=3, \end{aligned}\right\}$ to find the values of x and y.

$x=5,\ y=2.$

11. Given $\left\{\begin{aligned} 12x+8y&=116, \\ 2x-\ y&=3, \end{aligned}\right\}$ to find the values of x and y.

$x=5,\ y=\ 7.$

12. Given $\left\{\begin{aligned} \frac{x}{7}-\frac{7y}{10}&=-20, \\ \frac{x}{4}+3y&=134, \end{aligned}\right\}$ to find the values of x and y

$x=56,\ y=40.$

13. Given $\left\{\begin{aligned} ax+by&=c, \\ mx+ny&=d, \end{aligned}\right\}$ to find the values of x and y.

$x=\frac{bd-nc}{bm-an},\ y=\frac{cm-ad}{bm-an}$

14. Given $\left\{\begin{aligned} \frac{x}{15}-\frac{y}{20}&=1, \\ \frac{y}{10}-\frac{x}{30}&=1, \end{aligned}\right\}$ to find the values of x and y.

$x=30,\ y=20.$

15. Given $\left\{\begin{aligned} \frac{x}{2}+\frac{y}{3}&=7, \\ \frac{x}{3}+\frac{y}{2}&=8, \end{aligned}\right\}$ to find the values of x and y.

$x=6,\ y=12.$

16. Given $\left\{\begin{aligned} \frac{x+y}{2}&=5, \\ \frac{y-z}{4}&=1, \\ \frac{x-z}{6}&=\frac{1}{3}, \end{aligned}\right\}$ to find the values of x, y, and z.

$x=4,\ y=6,\ z=2.$

17. Given $\left\{\begin{aligned}\frac{6y-4x}{3z-7}&=1,\\ \frac{5z-x}{2y-3z}&=1,\\ \frac{y-2z}{3y-2x}&=1,\end{aligned}\right\}$ to find the values of x, y, and z.

$x=10,\ y=7,\ z=3.$

18. Given $\sqrt{\frac{x+1}{x-1}}+\sqrt{\frac{x-1}{x+1}}=a$, to find the values of x.

$x=\pm\frac{a}{a^2-4}\sqrt{a^2-4}.$

19. Given $\sqrt{a+x}+\sqrt{a-x}=b$, to find the values of x.

$x=\pm\frac{b}{2}\sqrt{4a-b^2}.$

20. Given $\sqrt{a(a+x)}=a-\sqrt{a(a-x)}$, to find the values of x.

$x=\pm\frac{1}{2}a\sqrt{3}.$

21. Given $\frac{2}{x+\sqrt{2-x^2}}+\frac{2}{x-\sqrt{2-x^2}}=x$, to find the values of x.

$x=\pm\sqrt{3}.$

22. Given $x\sqrt{6+x^2}=1+x^2$, to find the values of x.

$x=\pm\frac{1}{2}.$

23. Given $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{x}}-\sqrt{\frac{x}{b}}=0$, to find the values of x.

$x=\pm 2\sqrt{ab-b^2}.$

24. Given $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$, to find the values of x.

$x=4$, and -1.

25. Given $3x-\frac{3x-10}{9-2x}=2+\frac{6x^2-40}{2x-1}$, to find the values of x.

$x=11\frac{1}{2}$, and 4.

26. Given $x+4+\frac{7x-8}{x}=13$, to find the values of x.

$x=4$, and -2.

27. Given $14+4x-\frac{x+7}{x-7}=3x+\frac{9+4x}{3}$, to find the values of x.

$x=28$, and 9.

28. Given $\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1$, to find the values of x.

$x=21$, and 5.

29. Given $2x+18-\frac{8x^2+16}{4x+7}=27-\frac{12x-11}{2x-3}$, to find the values of x.

$x=8$, and 5.

30. Given $\frac{x^4+2x^3+8}{x^2+x-6}=x^2+x+8$, to find the values of x.

$x=4$, and $-4\frac{2}{3}$.

31. Given $x+\frac{\sqrt{x-3}}{2}=8$, to find x. $x=9\frac{1}{4}$, and 7.

32. Given $\sqrt{4+\sqrt{2x^3+x^2}}=\frac{x+4}{2}$, to find x.

$x=12$, and 4.

33. Given $x-\frac{12+8\sqrt{x}}{x-5}=0$. $x=9$, and 4.

Suggestion.—$x^2-5x=12+8\sqrt{x}$, whence $x^2-4x+4=16+8\sqrt{x}+x$, and $x-2=4+\sqrt{x}$.

$x=9$, and 4.

34. Given $\sqrt{a+x}+\sqrt{b+x}=\sqrt{a+b+2x}$, to find x.

$x=-a$, and $-b$.

35. Given $x+\sqrt{x}:x-\sqrt{x}::3\sqrt{x}+6:2\sqrt{x}$, to find x.

$x=9$, and 4.

36. Given $\frac{x+a}{x}+\frac{x}{x+a}=b$, to find x.

$$x=\frac{a}{2}\left(-1\pm\sqrt{\frac{b+2}{b-2}}\right).$$

37. Given $\left\{\begin{array}{l} x+y : x :: 3 : 2, \\ xy+y^2 = 3, \end{array}\right\}$ to find the values of x and y. $x = \pm 2, y = \pm 1.$

38. Given $\left\{\begin{array}{l} x+y : x :: 9 : 5, \\ xy = 20, \end{array}\right\}$ to find the values of x and y. $x = \pm 5, y = \pm 4.$

39. Given $\left\{\begin{array}{l} 2x^2+3xy = 44, \\ x-y : x :: 3 : 4, \end{array}\right\}$ to find the values of x and y. $x = \pm 4, y = \pm 1.$

40. Given $\left\{\begin{array}{l} \dfrac{x+y}{x-y} = 5, \\ x^2-y^2 = 5, \end{array}\right\}$ to find the values of x and y.

$x = \pm 3, y = \pm 2.$

41. Given $\left\{\begin{array}{l} \dfrac{2x-y}{2x+y} = \dfrac{3}{5}, \\ 3x^2-5xy = 8, \end{array}\right\}$ to find the values of x and y.

$x = \pm 4, y = \pm 2.$

42. Given $4x^2-2xy = 12$, $2y^2+3xy = 8$, to find x and y. $x = \pm 2$, and $\mp\frac{3}{7}\sqrt{7}$; $y = \pm 1$, and $\pm\frac{8}{7}\sqrt{7}$.

43. Given $3x^2+xy = 68$, and $4y^2+3xy = 160$, to find x and y.

$x = \pm 4$, and $\mp\frac{34}{9}\sqrt{3}$; $y = \pm 5$, and $\pm\frac{48}{9}\sqrt{3}$.

44. Given $2x^2-3xy+y^2 = 4$, and $2xy-3y^2-x^2 = -9$, to find x and y.

$x = \pm 3$, and $\mp\frac{1}{6}\sqrt{2}$; $y = \pm 2$, and $\pm\frac{7}{6}\sqrt{2}$.

45. Given $x^2+xy = 12$, and $xy-2y^2 = 1$, to find x and y. $x = \pm 3$, and $\pm\frac{4}{3}\sqrt{6}$; $y = \pm 1$, and $\pm\frac{1}{6}\sqrt{6}$.

46. Given $x^3y-y = 21$, and $x^2y-xy = 6$, to find x and y. $x = 2$, and $\frac{1}{2}$; $y = 3$, and -24.

47. Given $x^2+x+y = 18-y^2$, and $xy = 6$, to find x and y.
$x = 2$, and 3, or $-3\mp\sqrt{3}$; $y = 3$, and 2, or $-3\pm\sqrt{3}$.

48. Given $\frac{x^2}{y}+\frac{y^2}{x} = a$, and $x+y = 2b$, to find x and y.
(Put $x = z+v$, and $y = z-v$.)

$$x = b\left(1\pm\sqrt{\frac{a-2b}{a+6b}}\right);\ y = b\left(1\mp\sqrt{\frac{a-2b}{a+6b}}\right).$$

49. Given $x^2+x = \frac{12}{y}$, and $x^3y+y = 18$, to find x and y.
$x = 2$, and $\frac{1}{2}$; $y = 2$, and 6.

50. Given $x^2+4y^2 = 256-4xy$, and $4y^2-x^2 = 64$, to find x and y. $x = \pm 6$; $y = \pm 5$.

THE END.